Desanka Radunović – NUMERIČKE METODE

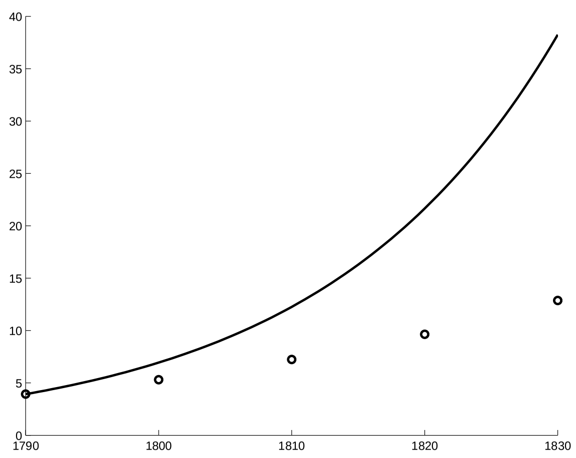
Diferencijalne jednačine

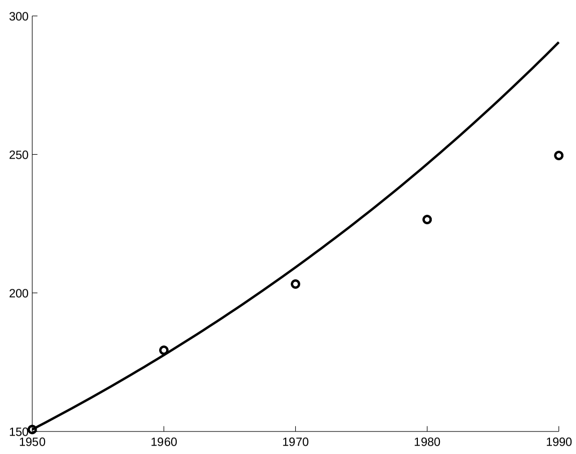
Cauchy-jevi problemi (početnih vrednosti)

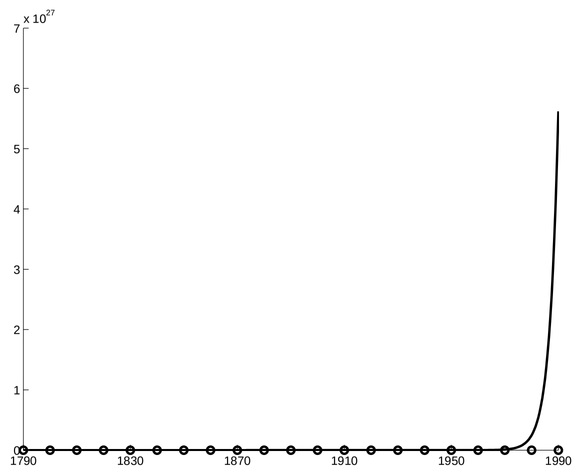
◇ *Malthys-ov populacioni model* (linearan)

$\beta =$stopa rađanja, $\delta =$stopa umiranja, $\gamma = \beta - \delta =$stopa priraštaja

$$u(t+\Delta t) = u(t) + (\beta - \delta)\, u(t)\, \Delta t \quad \xrightarrow{\Delta t \to 0} \quad \frac{d}{dt} u(t) = \gamma\, u(t), \quad u(0) = u_0$$







Modelovanje broja stanovnika SAD osnovnim populacionim modelom

◇ *Logistički populacioni model* (Pierre Verhulst, 1838) (nelinearan)

$$\beta(t) = \beta_0 - \beta_1 u(t), \quad \delta(t) = \delta_0 + \delta_1 u(t), \qquad \beta_1,\, \delta_1 > 0$$

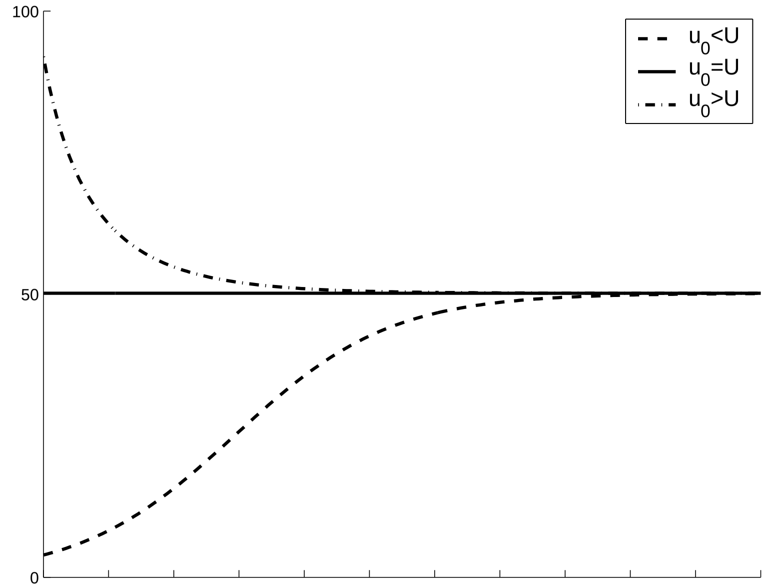
$$\gamma(t) = (\beta_0 - \delta_0) - (\beta_1 - \delta_1)\, u(t) = \gamma_0 \left(1 - \frac{1}{U}\, u(t)\right)$$

γ_0 maksimalni priraštaj

U optimalni broj jedinki

$$\frac{d}{dt}\, u(t) = \gamma_0 \left(1 - \frac{1}{U}\, u(t)\right) u(t),$$
$$u(0) = u_0$$

$$u(t) = \frac{u_0\, U}{u_0 + (U - u_0)\, e^{-\gamma_0 t}}$$



logistička kriva

◇ *Lotka-Volterra model grabljivice i plena* (nelinearan sistem)

biljojedi $$\frac{d}{dt}u_1(t) = \big(a_1 - b_1\, u_2(t)\big)\, u_1(t), \qquad a_1,\, b_1 > 0$$

mesojedi $$\frac{d}{dt}u_2(t) = \big(-a_2 + b_2\, u_1(t)\big)\, u_2(t), \qquad a_2,\, b_2 > 0$$

smena $$u_k(x) \equiv u^{(k)}(x), \qquad k = 0, \ldots, m-2,$$

$$u^{(m)}(x) = f(x, u, u', \ldots, u^{(m-1)}) \qquad \longrightarrow \qquad \begin{aligned} u_k'(x) &= u_{k+1}(x) \\ u_{m-1}' &= f(x, u_0, \ldots, u_{m-1}) \end{aligned}$$

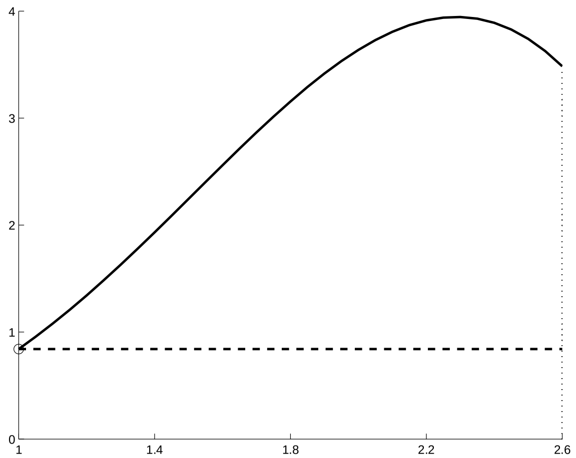
$$\boxed{u_k'(x) = f_k(x, u_1, \ldots, u_m), \qquad u_k(x_0) = u_{0k}, \qquad k = 1, \ldots, m.}$$

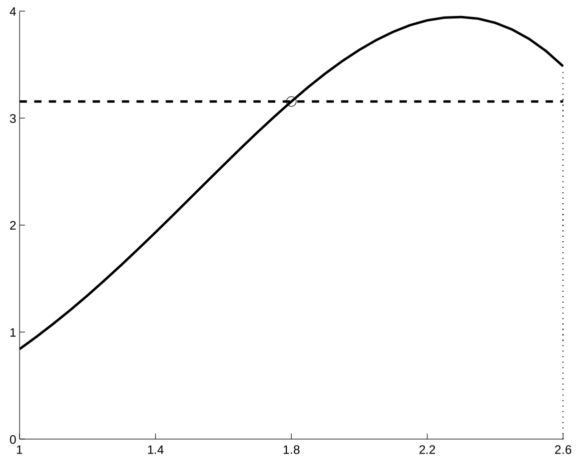
◇ Kamata $$v_{n+1} = v_n + h\, r\, v_n, \quad v_0 = u(0) \qquad u'(t) = r\, u(t)$$

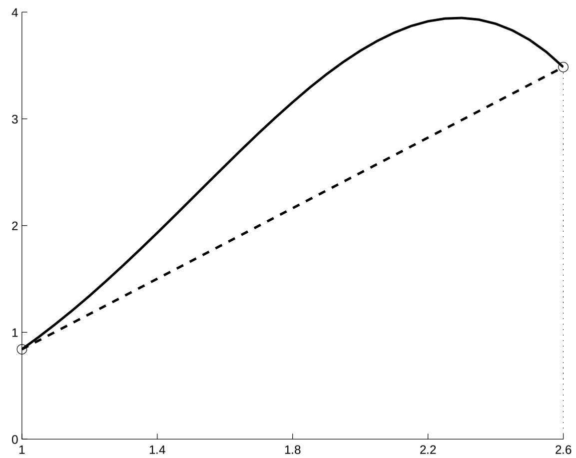
$$v_{n+1} = v_n + h\, f\big(t_{n+1/2}, v_{n+1/2}\big),$$

$$v_{n+1} = v_n + \frac{h}{2}\Big(f(t_n, v_n) + f\big(t_{n+1}, v_{n+1}\big)\Big)$$

Kvadraturne formule primenjene u *Euler-ovoj* metodi i njenim modifikacijama







Metode tipa Runge–Kutta

$$u(x+h) \approx v(x+h) \equiv u(x) + \sum_{i=1}^{n} c_i k_i(h)$$

$$k_1(h) = hf(x,u) \qquad\qquad (0 = \alpha_1 \le \alpha_2 \le \cdots \le \alpha_n \le 1)$$

$$k_2(h) = hf\big(x + \alpha_2 h, u + \beta_{21} k_1(h)\big)$$

$$\vdots$$

$$k_n(h) = hf\big(x + \alpha_n h, u + \beta_{n1} k_1(h) + \cdots + \beta_{n,n-1} k_{n-1}(h)\big)$$

Za $\quad \epsilon(0) = \epsilon'(0) = \cdots = \epsilon^{(p)}(0) = 0 \quad$ metoda je reda $\quad p$

$$\epsilon(h) = u(x+h) - v(x+h) = \frac{\epsilon^{(p+1)}(\theta h)}{(p+1)!} h^{p+1}$$

$\boxed{n = 1}$ $\quad p = 1$

Euler-ova metoda $\quad v(x + h) = v(x) + hf(x, v(x))$

$\boxed{n = 2}$ $\quad p = 2$

Euler-trapez

$$k_1 = hf(x, v(x)), \qquad k_2 = hf(x + h,\, v(x) + k_1)$$
$$v(x + h) = v(x) + \frac{h}{2}(k_1 + k_2)$$

Euler-pravougaonik

$$k_1 = hf(x, v(x)), \qquad k_2 = hf(x + \tfrac{h}{2}, v(x) + \tfrac{k_1}{2})$$
$$v(x + h) = v(x) + k_2$$

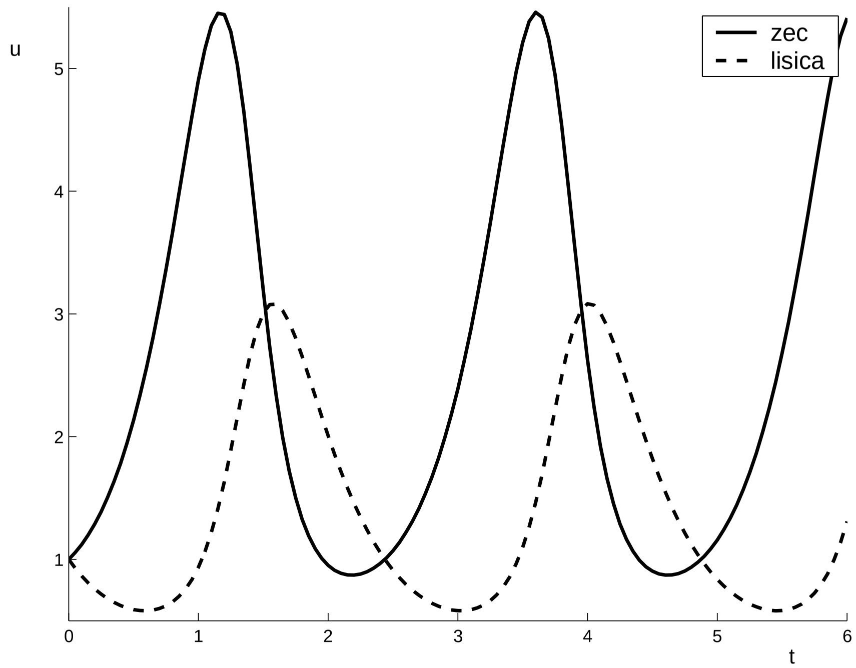
$\boxed{n = 4}$ $\quad p = 4$

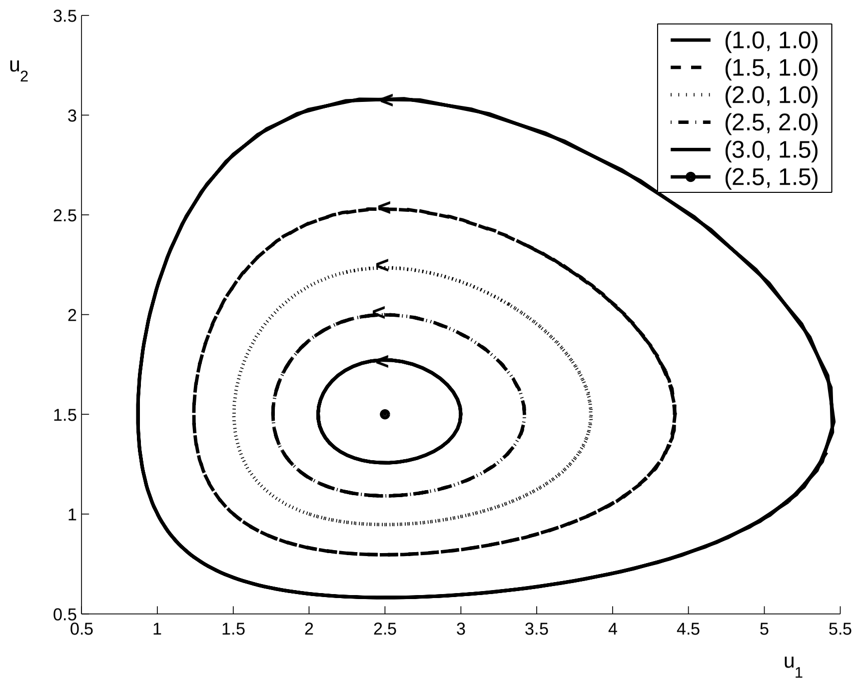
Runge-Kutta

$$k_1 = hf(x, v(x)), \qquad k_2 = hf(x + \tfrac{h}{2}, v(x) + \tfrac{k_1}{2}),$$
$$k_3 = hf(x + \tfrac{h}{2}, v(x) + \tfrac{k_2}{2}), \qquad k_4 = hf(x + h, v(x) + k_3),$$
$$v(x + h) = u(x) + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

Rungeova ocena greške $u(x+2h) - v_h(x+2h) \approx \frac{v_h(x+2h) - v_{2h}(x+2h)}{2^p - 1}$

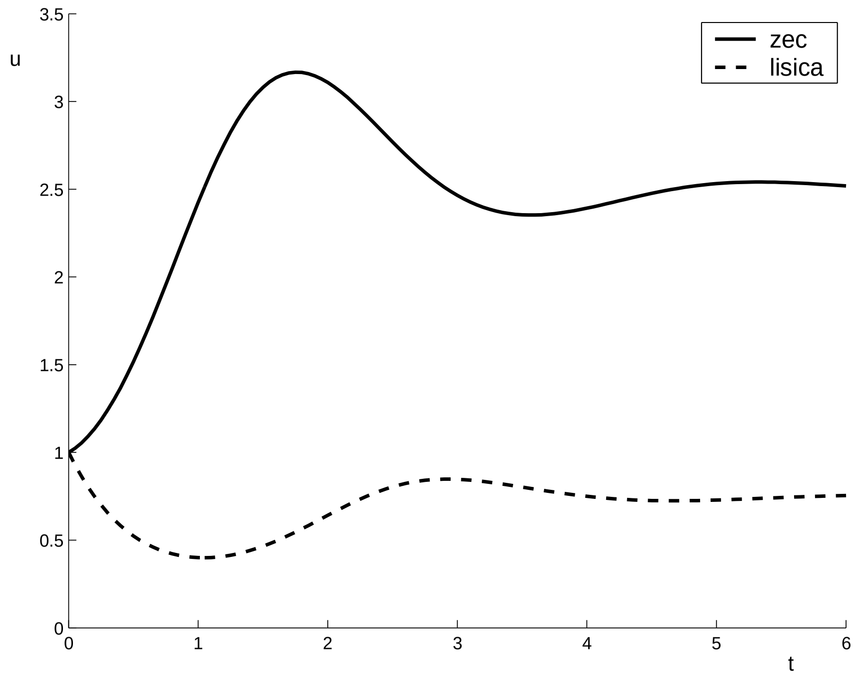
◇ Promena broja zečeva i lisica u vremenu (levo), i predstavljanje njihove zavisnosti u faznoj ravni za različite izbore početnog stanja (desno)

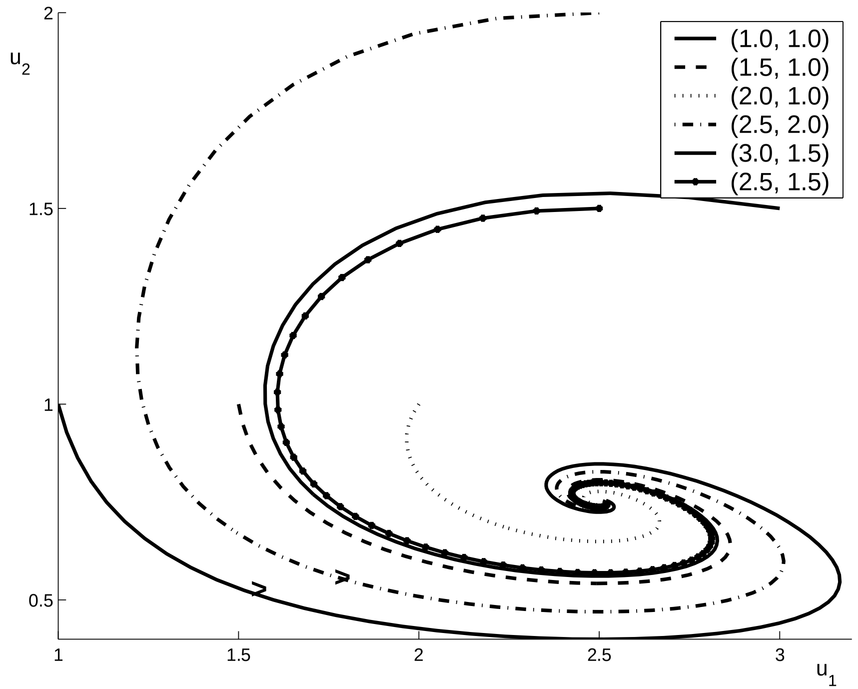




◇ Model zečeva i lisica sa definisanim optimalnim brojem zečeva U

$$u_1'(t) = \big(3(1 - \frac{u_1(t)}{U}) - 2u_2(t)\big)\, u_1(t), \qquad u_1(0) = 1$$
$$u_2'(t) = \big(-2.5 + u_1(t)\big)\, u_2(t), \qquad u_2(0) = 1$$





Prediktor–korektor metode (višeslojne)

$$\sum_{i=0}^{n} a_i v_{j-i} - h \sum_{i=0}^{n} b_i f(x_{j-i}, v_{j-i}) = 0,$$

$b_0 = 0$ ekstrapolacione

$b_0 \neq 0$ interapolacione

Milne-ova metoda $p = 4$

$$v_j^* = v_{j-4} + \frac{4h}{3}(2f_{j-1} - f_{j-2} + 2f_{j-3})$$
$$v_j = v_{j-2} + \frac{h}{3}(f_j^* + 4f_{j-1} + f_{j-2})$$

$$\varepsilon \approx \frac{1}{29}|v - v^*|$$

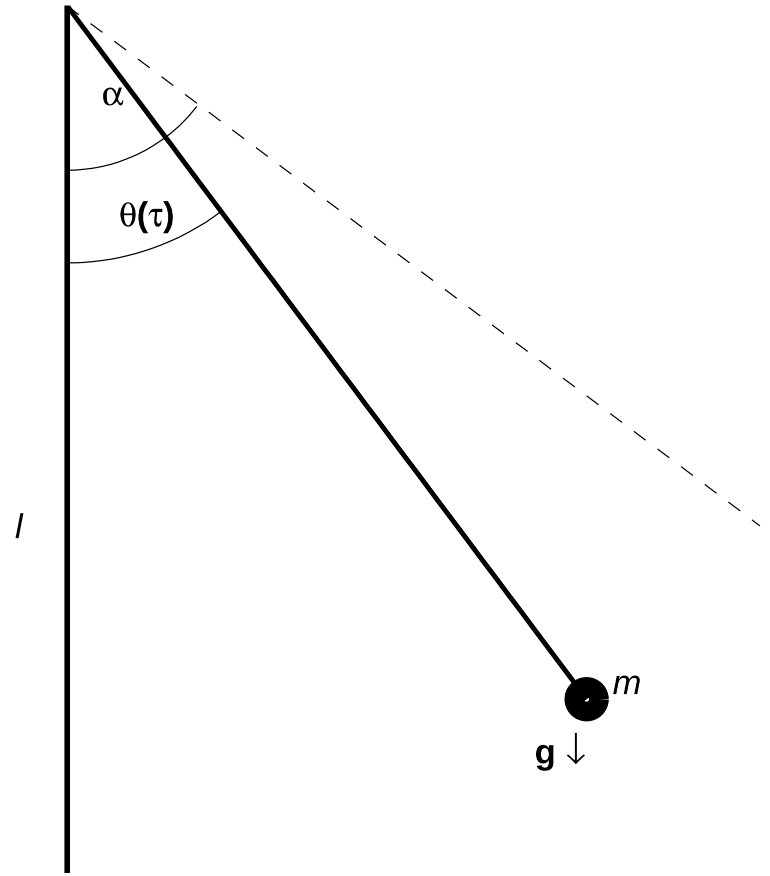
Adams-ova metoda $p = 4$

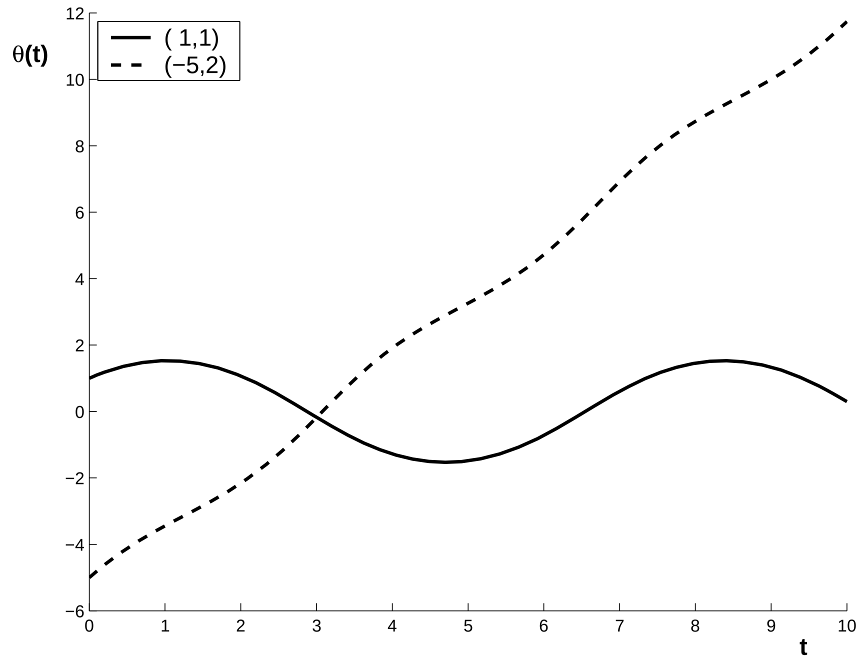
$$v_j^* = v_{j-1} + h(f_{j-1} + \frac{1}{2}\Delta f_{j-2} + \frac{5}{12}\Delta^2 f_{j-3} + \frac{3}{8}\Delta^3 f_{j-4})$$
$$v_j = v_{j-1} + h(f_j^* - \frac{1}{2}\Delta f_{j-1}^* - \frac{1}{12}\Delta^2 f_{j-2}^* - \frac{1}{24}\Delta^3 f_{j-3}^*)$$

$$\varepsilon \approx \frac{1}{14}|v - v^*|$$

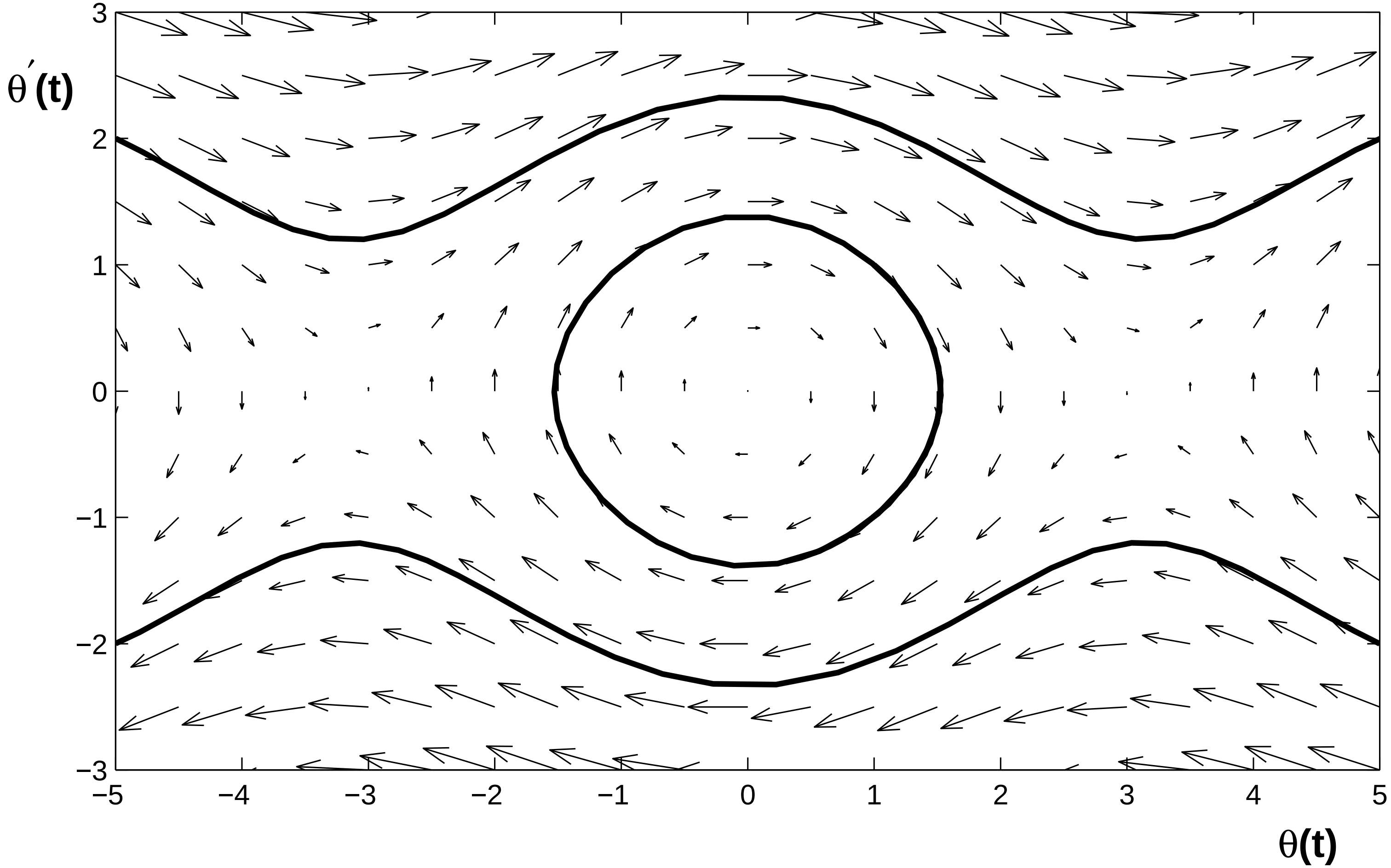
◇ Matematičko klatno

$$\theta''(t) + \sin\theta(t) = 0, \qquad \theta(0) = \alpha, \quad \theta'(0) = 0, \quad \longrightarrow \quad u_1'(t) = u_2(t), \quad u_1(0) = \alpha, \qquad u_2'(t) = -\sin u_1(t), \quad u_2(0) = 0.$$





Fazna ravan matematičkog klatna



Stabilnost numeričkog rešenja jednačine

$$u'(t) = a\,u(t), \quad u(0) = 1, \qquad \text{za} \qquad a < 0$$

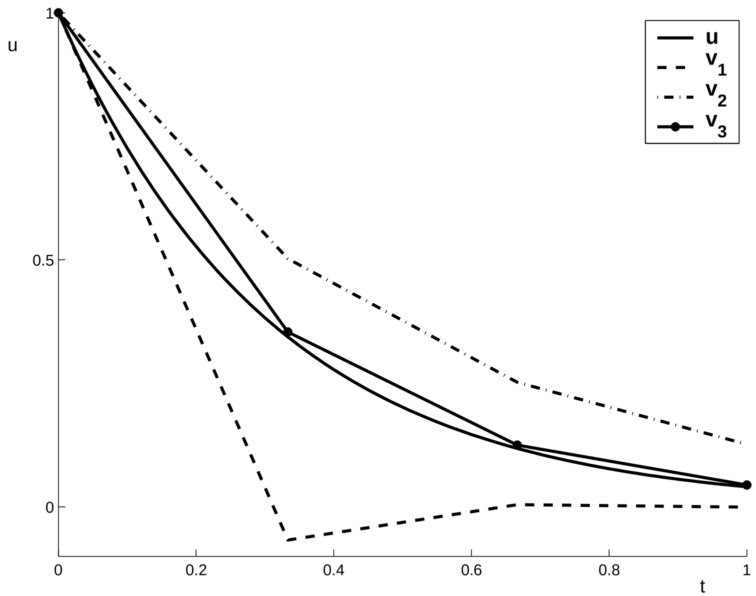
Euler $$v_{1,n} = (1 + h\,a)^n$$

Euler – 1.modifikacija $$v_{2,n} = \big(1 + h\,a + \frac{h^2a^2}{4}\big)^n,$$

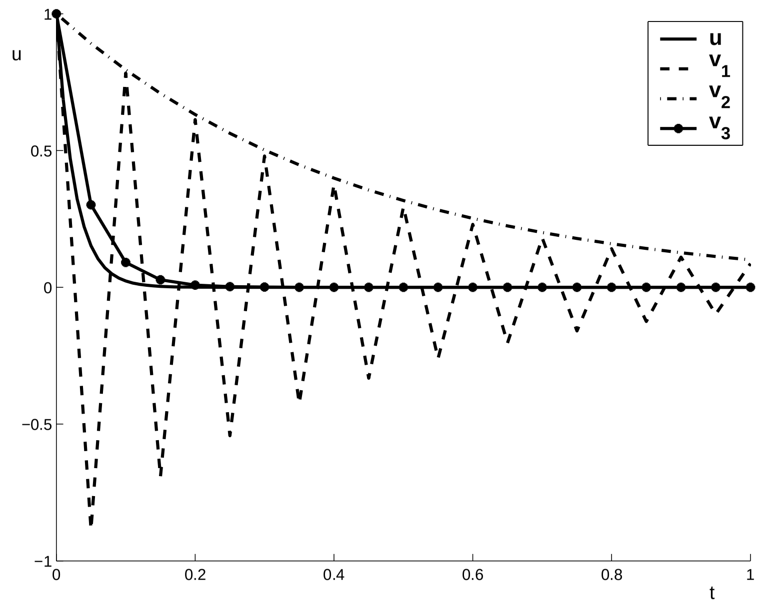
Runge – Kutta $$v_{3,n} = \big(1 + h\,a + \frac{h^2a^2}{2} + \frac{h^3a^3}{6} + \frac{h^4a^4}{24}\big)^n$$

Euler – 2.modifikacija $$v_{4,n} = \left(\frac{2 + h\,a}{2 - h\,a}\right)^n$$

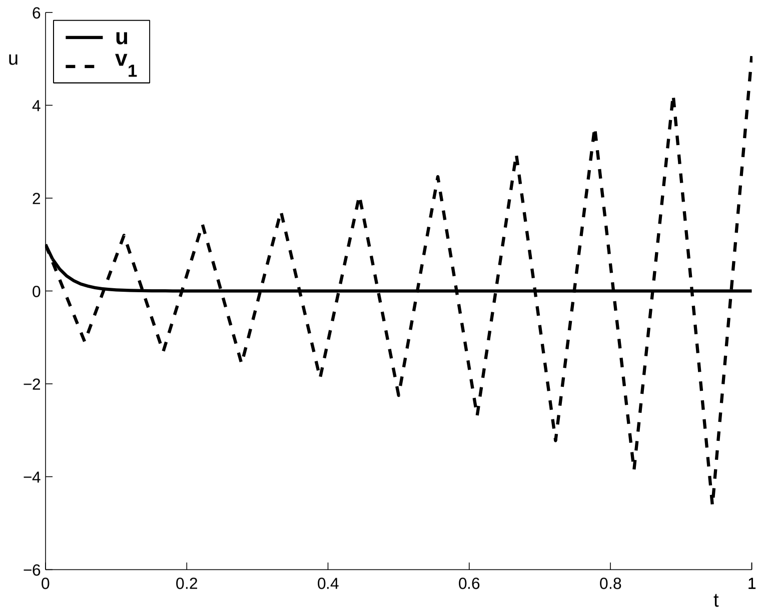
◇ Primeri: $a = -3.2$ (slika gore levo), $a = -37.7$ (ostale slike)



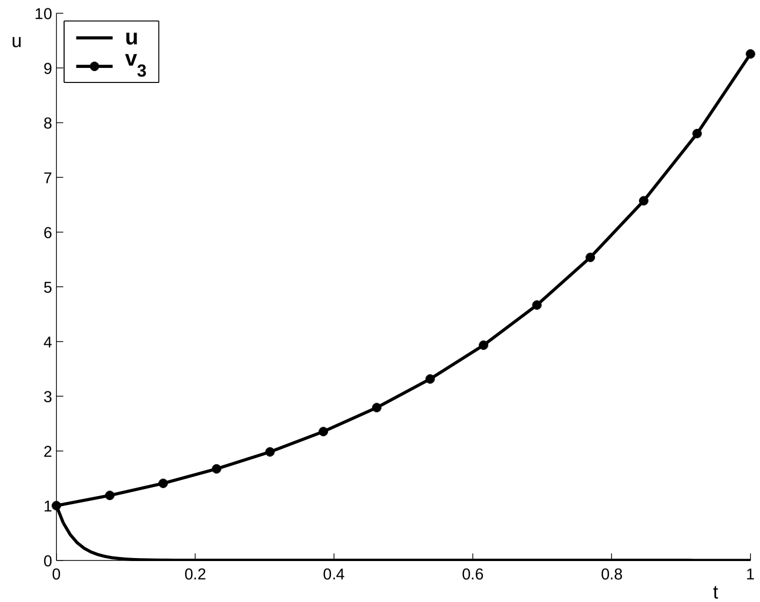
h=1/3



h=1/20



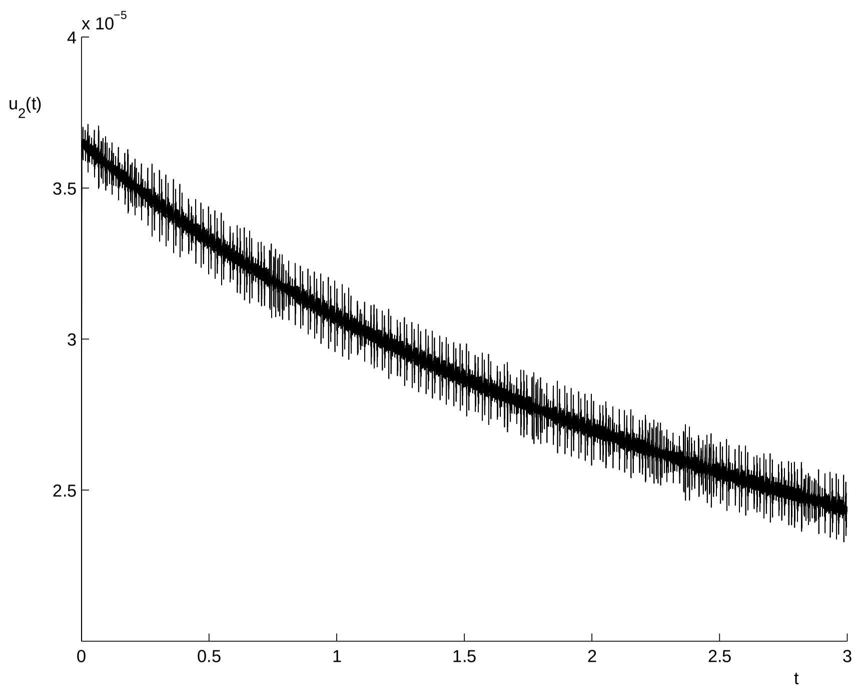
Euler, h=1/18



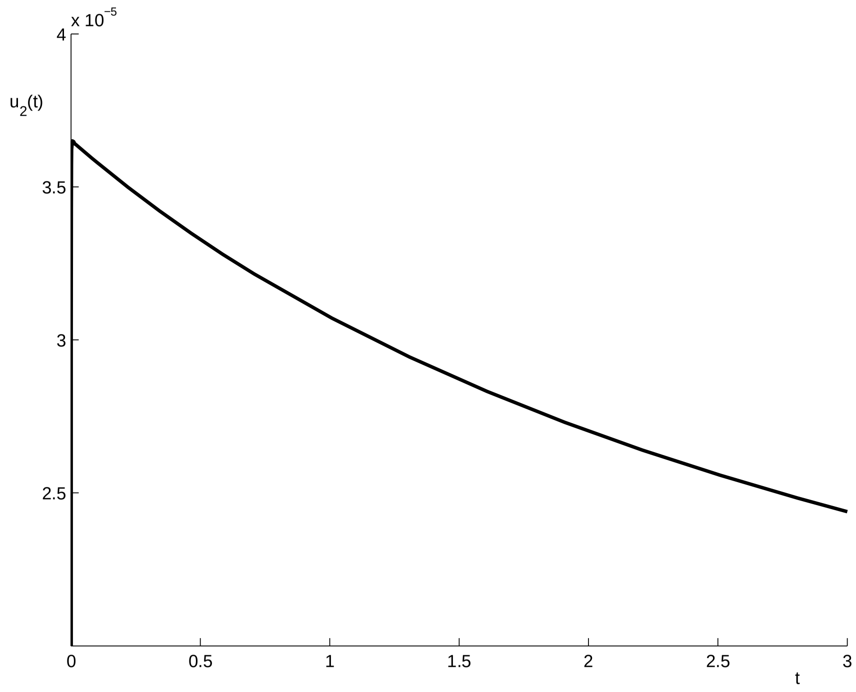
Runge−Kutta, h=1/13

◇ Krut sistem kojim se modeluje reakcija između tri hemikalije

$$u_1'(t) = -0.04u_1(t) + 10^4 u_2(t)\, u_3(t)$$
$$u_2'(t) = 0.04u_1(t) - 10^4 u_2(t)\, u_3(t) - 3 \cdot 10^7 u_2(t)^2$$
$$u_3'(t) = 3 \cdot 10^7 u_2(t)^2$$



ode45



ode15s

metoda Runge-Kutta

višeslojna metoda za krute sisteme

Aproksimativne metode

Metoda uzastopnih aproksimacija

$$u_0(x) = u_0 \qquad u_{n+1}(x) = u_0 + \int_{x_0}^{x} f(t, u_n(t))\, dt, \quad n = 0, 1, \ldots .$$

Metoda Taylorovog razvoja $\qquad u_n(x) = \sum_{j=0}^{n} \frac{u^{(j)}(x_0)}{j!}(x - x_0)^j$

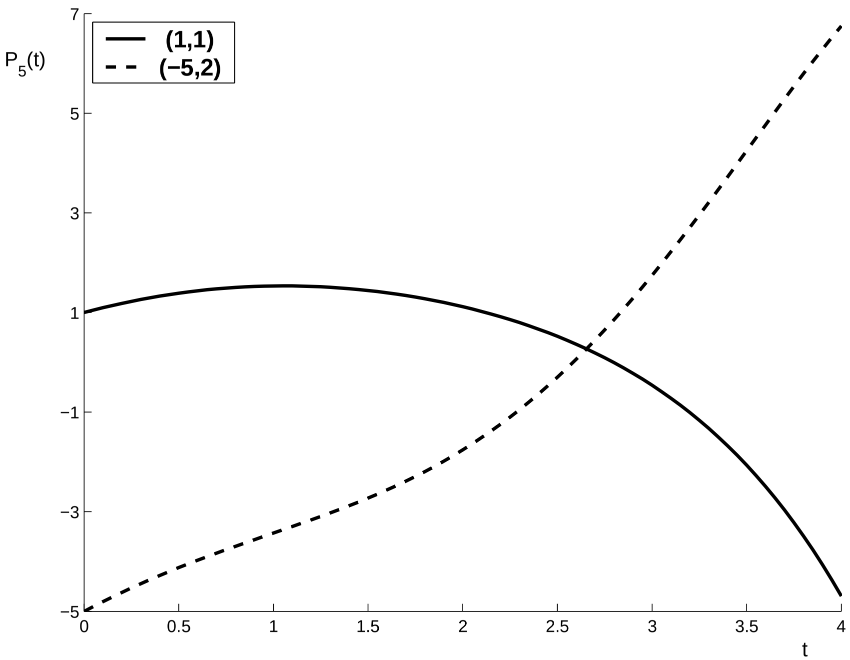
Metoda stepenih redova $\qquad u''(x) + p(x)u'(x) + q(x)u(x) = f(x)$

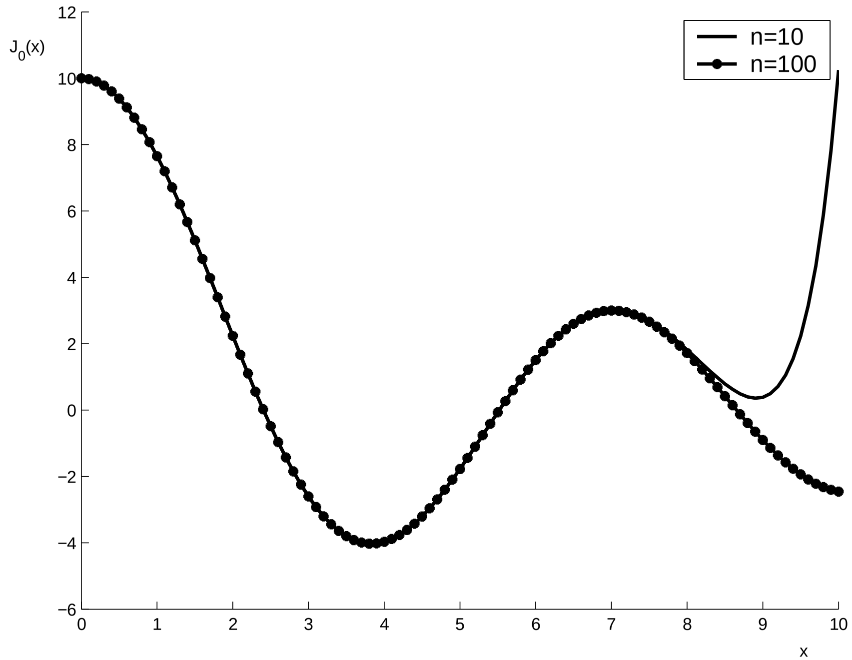
$$\sum_{j=2}^{\infty} j(j-1)c_j x^{j-2} + \Big(\sum_{j=0}^{\infty} p_j x^j\Big)\Big(\sum_{j=1}^{\infty} j c_j x^{j-1}\Big) + \Big(\sum_{j=0}^{\infty} q_j x^j\Big)\Big(\sum_{j=0}^{\infty} c_j x^j\Big) = \sum_{j=0}^{\infty} f_j x^j$$

◇

$$\theta''(t) + \sin\theta(t) = 0,$$
$$\theta(0) = \alpha, \quad \theta'(0) = 0,$$

$$x^2 \frac{d^2u}{dx^2} + x\frac{du}{dx} + x^2 u = 0$$





Taylor-ov razvoj, Aproksimacija Bessel-ove funkcije $J_0(x)$

Granični problemi

◇ Raspodela toplote u tankom štapu, koji je izolovan na desnom kraju, a na čijem se levom kraju održava konstantna temperatura

$$u''(x) - c\,u(x) = 0, \quad x \in [0,1], \qquad u(0) = 1, \quad u'(1) = 0$$

◇ Deformacija grede, koja se u četiri tačke greda oslanja na nosače, pod uticajem spoljašnje sile $f(x)$

$$u^{(4)}(x) = f(x), \qquad 0 \le x \le 1,$$
$$u(x_i) = 0, \quad i = 1,2,3,4, \qquad 0 \le x_1 < x_2 < x_3 < x_4 \le 1$$

$$-u''(x) + q(x)u(x) = f(x), \qquad 0 \le x \le 1$$

$$\alpha_1 u'(0) + \beta_1 u(0) = 0, \qquad \alpha_2 u'(1) + \beta_2 u(1) = 0$$

Dirichlet-ovi gran. uslovi $\qquad u(0) = u(1) = 0,$

Neumann-ovi gran. uslovi $\qquad u'(0) = u'(1) = 0$

mešoviti gran. uslovi $\qquad u'(0) - \sigma_1 u(0) = 0, \qquad u'(1) + \sigma_2 u(1) = 0$

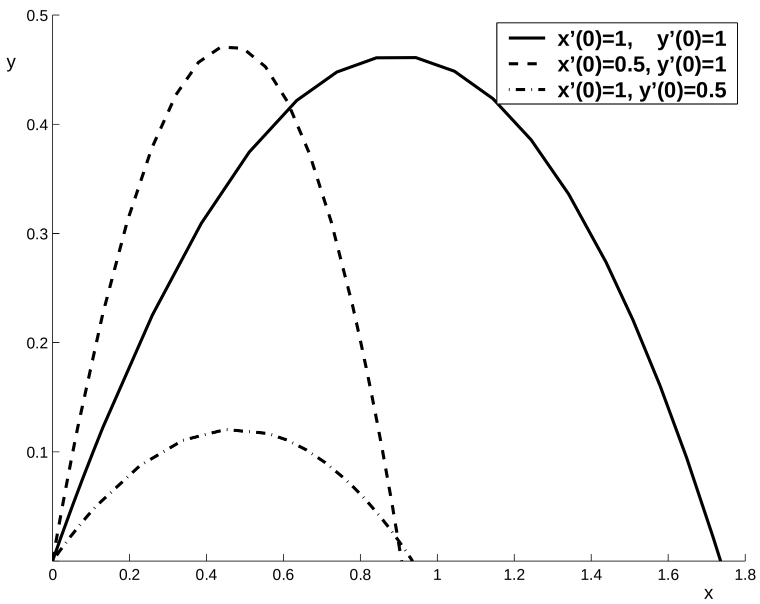
♠ Ako su funkcije $q(x)$, $f(x) \in \mathcal{C}[0,1]$ i $q(x) \ge 0$, tada Dirichlet-ov granični problem ima jedinstveno rešenje $u(x) \in \mathcal{C}^2[0,1]$.

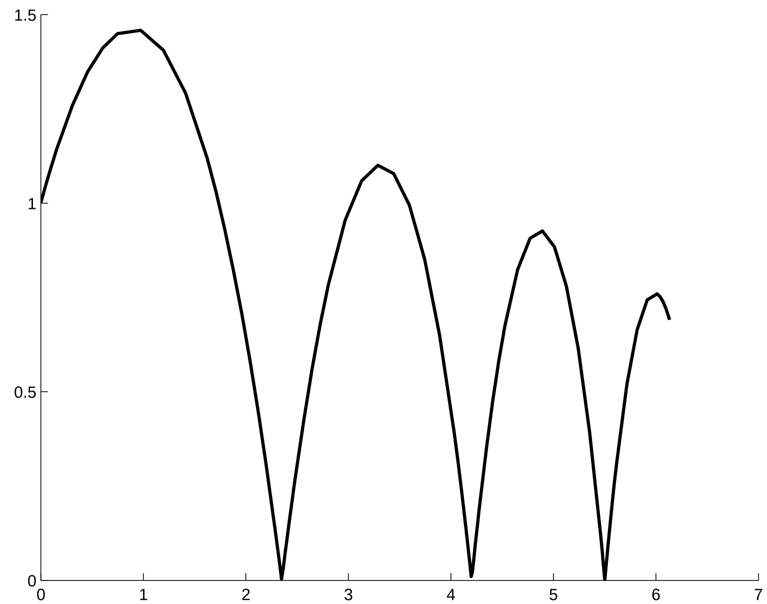
♠ Ako su funkcije $q(x)$, $f(x) \in \mathcal{C}[0,1]$, $q(x) \ge 0$ i $\sigma_1 > 0$, $\sigma_2 > 0$, tada mešoviti granični problem ima jedinstveno rešenje $u(x) \in \mathcal{C}^2[0,1]$.

Metoda gađanja

◇

$$x''(t) = -\frac{c}{m}\sqrt{\big(x'(t)\big)^2 + \big(y'(t)\big)^2}\,x'(t), \qquad x(0)=0, \quad x(T)=D,$$
$$y''(t) = -\frac{c}{m}\sqrt{\big(x'(t)\big)^2 + \big(y'(t)\big)^2}\,y'(t) - g, \qquad y(0)=0, \quad y(T)=0.$$





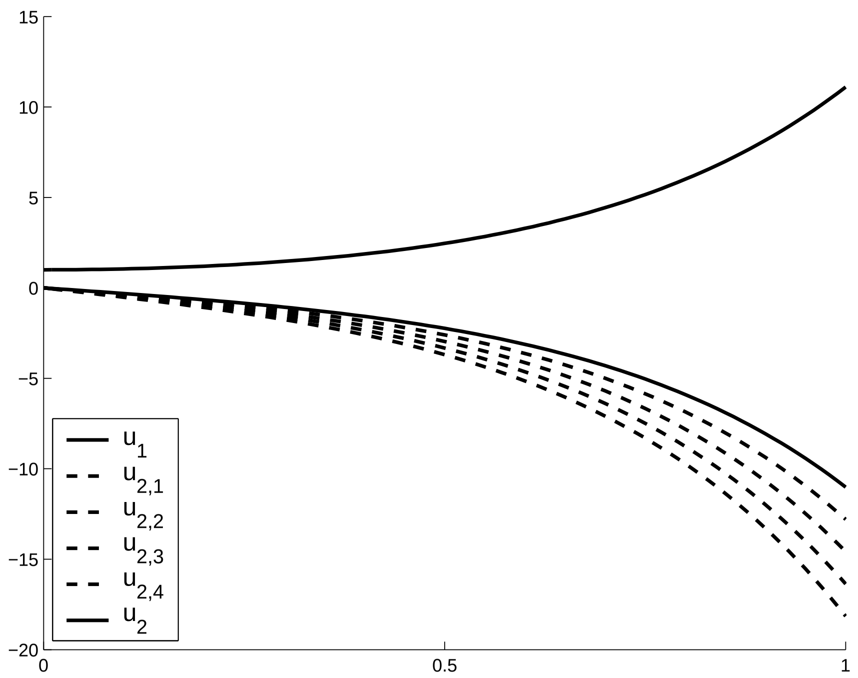
Putanja projektila, lopte

◇ Raspodela temperature u štapu određena metodom gađanja (linearan problem)

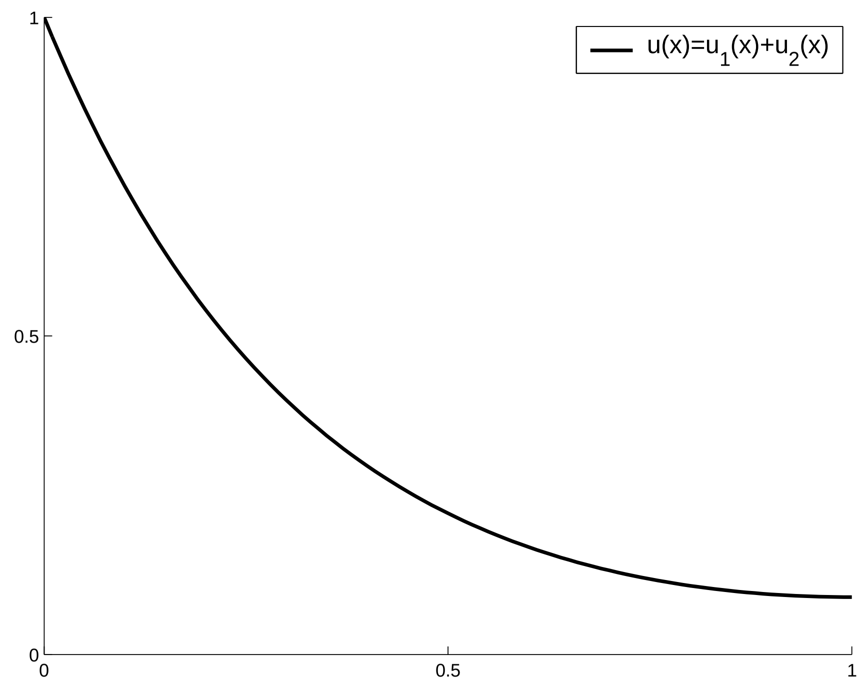
$$u(x) = u_1(x) + \gamma\, u_2(x)$$

$$u_1''(x) - c\, u_1(x) = 0, \qquad u_2''(x) - c\, u_2(x) = 0$$

$$u_1(0) = 1,\ u_1'(0) = 0 \qquad u_2(0) = 0,\ u_2'(0) = 1$$



funkcije $u_1(x)$ i $\gamma\, u_2(x)$



rešenje $u(x)$

Metoda konačnih razlika (metoda mreže)

mreža $\bar{\omega}_h = \{\, x_i \,|\, x_i = ih,\ i = 0, \ldots, n,\ h = \frac{1}{n} \,\}$ $\qquad v_i \approx u(x_i)$

Aproksimacije izvoda količnicima konačnih razlika

$$v_{x,i} = \frac{1}{h}(v_{i+1} - v_i), \qquad v_{\bar{x},i} = \frac{1}{h}(v_i - v_{i-1})$$

$$v_{\dot{x},i} = \frac{1}{2h}(v_{i+1} - v_{i-1}) = \frac{1}{2}(v_{x,i} + v_{\bar{x},i})$$

za u'

$$v_{\bar{x}x,i} = \frac{1}{h^2}(v_{i+1} - 2v_i + v_{i-1}) = \frac{1}{h}(v_{x,i} - v_{\bar{x},i})$$

za u''

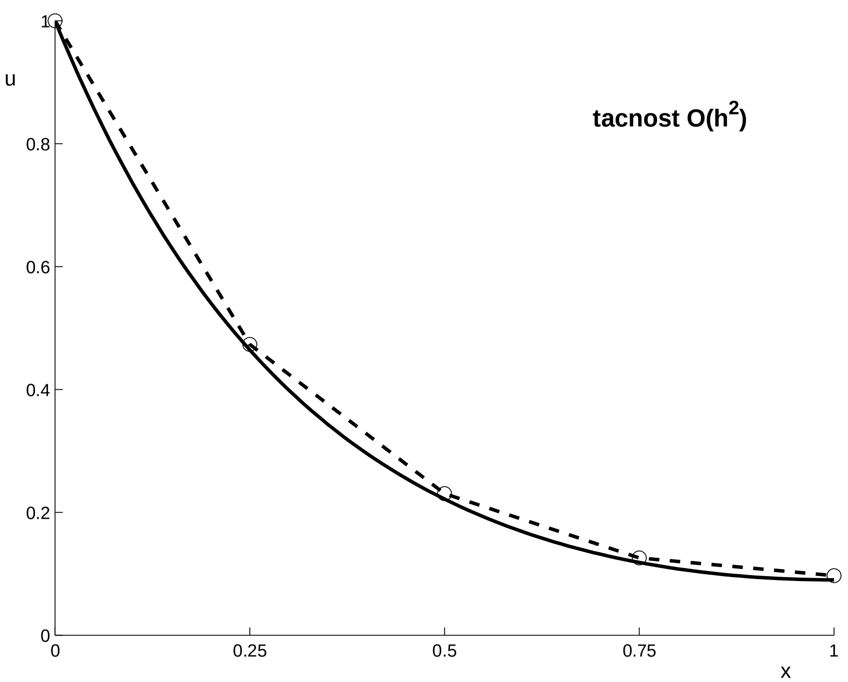
$$-v_{\bar{x}x,i} + q_i v_i = f_i, \qquad i = 1, \ldots, n-1,$$

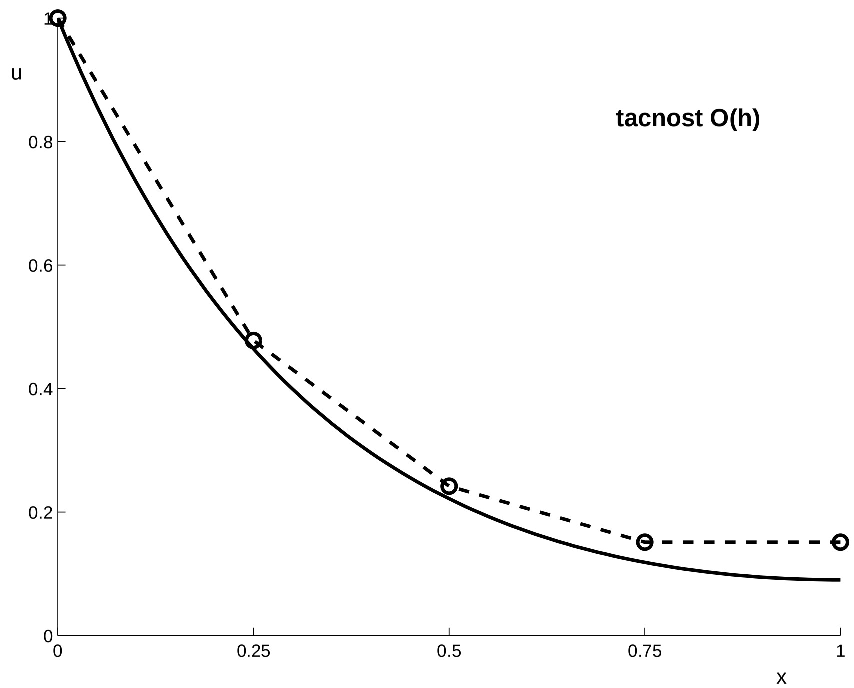
diferencijska šema

$$v_0 = v_n = 0 \qquad \text{ili} \qquad \begin{aligned} v_{x,0} - \sigma_1 v_0 &= 0, \\ v_{\bar{x},n} + \sigma_2 v_n &= 0 \end{aligned}$$

◇ $$u_{\bar{x},n} = \frac{1}{h}\big(u(x_n) - u(x_{n-1})\big) = u'(x_n) - \frac{h}{2}\,u''(x_n) + O(h^2)$$

$$v_0 = 1, \qquad v_{i-1} - (2 + c\,h^2)\,v_i + v_{i+1} = 0, \qquad v_{n-1} - \Big(1 + \frac{c\,h^2}{2}\Big)\,v_n = 0$$





Uticaj tačnosti aproksimacije graničnog uslova na tačnost rešenja

Metoda konačnih razlika za višedimenzione granične probleme

Poisson-ova jednačina (Laplace-ova jednačina za $f(x,y) \equiv 0$)

$$\Delta u(x,y) \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y), \quad (x,y) \in \Omega$$

$$u(x,y) = g(x,y), \quad (x,y) \in \text{gr}\,\Omega$$

mreža $\bar{\omega}_{h,k} = \{\,(x_i, y_j)\,|\,x_i = ih,\ y_j = jk,\}$ $\qquad v_{i,j} \approx u(x_i, y_j)$

Diferencijska šema ("krst šema")

$$v_{\bar{x}x,i,j} + v_{\bar{y}y,i,j} = f(x_i, y_j) \qquad (x_i, y_j) \in \omega_{h,k}$$

$$v_{i,j} = g(x_i, y_j) \qquad (x_i, y_j) \in \text{gr}\,\bar{\omega}_{h,k}$$

◇ $\bar{\Omega} = \{(x,y) \,\big|\, |x| \le 1,\ |y| \le 1,\ |x-y| \le 1\}$

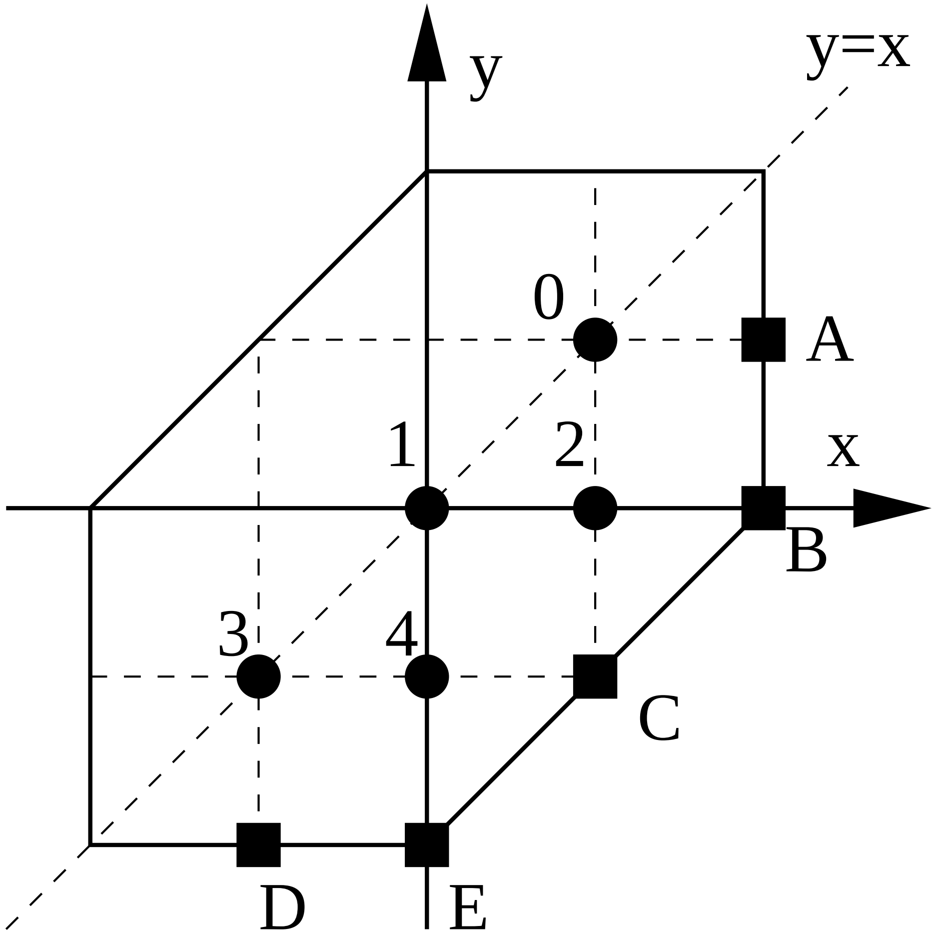
$$\Delta u(x,y) = x^2 + y^2, \quad (x,y) \in \Omega, \qquad u(x,y) = |x| + |y|, \quad (x,y) \in \text{gr}\,\Omega$$

○ unutrašnji čvorovi

$$v_{\bar{x}x,i,j} + v_{\bar{y}y,i,j} = x_i^2 + y_j^2,$$
$$i,j = 1,\ldots,n-1,$$

◇ granični čvorovi

$$v_{i,j} = |x_i| + |y_j|,$$
$$i = 0 \lor j = 0 \lor i = n \lor j = n$$



Krivolinijska granica $\bar{\Omega} = \{(x,y) \,|\, x^2 + y^2 \leq 1\}$

$$4\,(4v_1 - 4v_0) = 0$$

○ unutrašnji čvorovi

$$4\,(v_A - 2v_1 + v_0) + 4\,(v_2 - 2v_1 + v_2) = 1/4$$

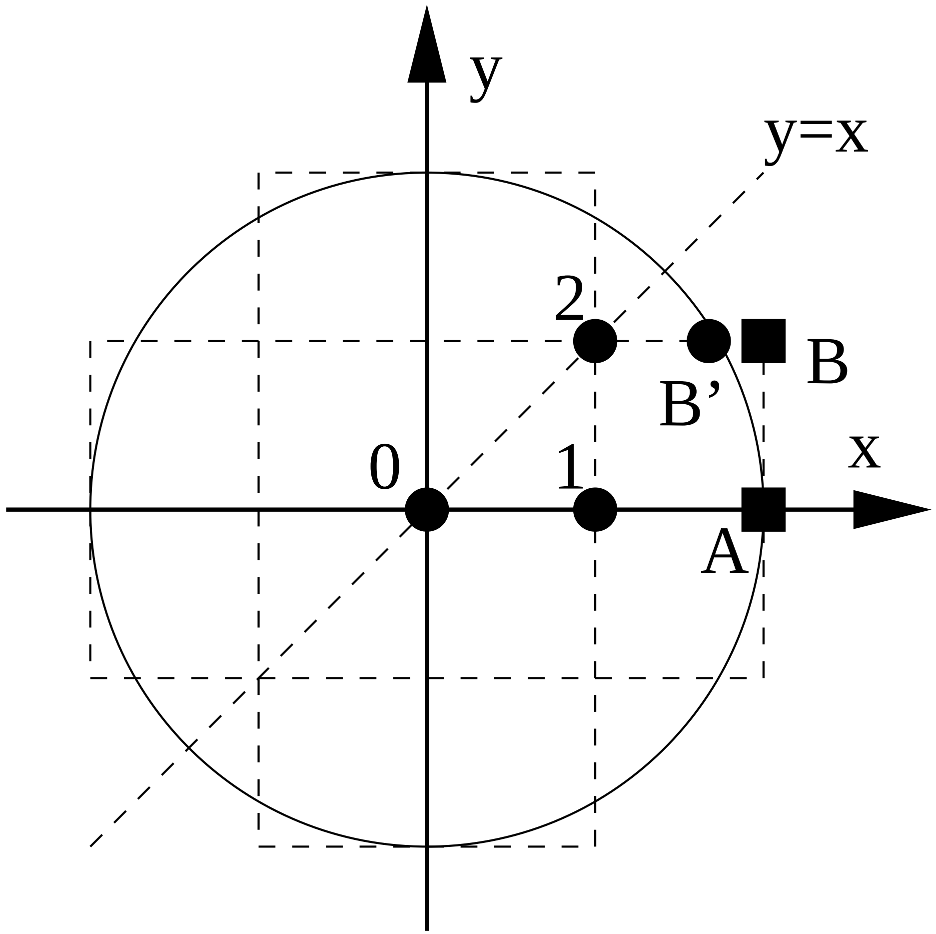
$$4\,(v_B - 2v_2 + 4\,v_1) + (v_B - 2v_2 + v_1) = 1/2$$

◇ granični čvorovi
(ekstrapolacija)

$$v_A = 1$$

$$v_B = -\frac{\delta}{h-\delta} v_2 + \frac{h}{h-\delta} v_{B'}$$

$$\delta = 1 - \sqrt{3}/2$$



$$P_{i+(j-1)\,(n-1)} \equiv (x_i,\, x_j)^T \qquad i,\, j = 1 \;\ldots, n-1.$$

$$\boxed{A\,\mathbf{v} = \mathbf{f}}$$

$$v_l = v(P_l),$$
$$f_l = f(P_l)$$
$$l = 1, \ldots, (n-1)^2$$

$$A = \frac{1}{h^2}\begin{pmatrix}
4 & -1 & & & -1 & & & & & \\
-1 & 4 & \ddots & & & -1 & & & & \\
 & \ddots & \ddots & & & & & & & \\
 & & & 4 & & & & & & \\
-1 & & & & 4 & -1 & & & & \\
 & -1 & & & -1 & 4 & & & & \\
 & & \ddots & & & \ddots & & & \ddots & \\
 & & & & & & & & & -1 \\
 & & & & & & 4 & -1 & & \\
 & & & & & & -1 & 4 & \ddots & \\
 & & & & & & & \ddots & \ddots & \\
 & & & & & & & & & 4
\end{pmatrix}$$

$$-\frac{1}{h^2}\,(v_{l-n+1}$$
$$+v_{l-1} - 4\,v_l + v_{l+1}$$
$$+\,v_{l+n-1}) = f_l$$

Varijacione metode

♠ L je samokonjugovan, pozitivno definisan linearni operator

$$L\,u = f \qquad \Longleftrightarrow \qquad I(u) = \min_w I(w), \quad I(w) = (Lw,\, w) - 2(f,\, w)$$

$$v(x) = \sum_{i=1}^{n} c_i \phi_i(x), \qquad R(x; c_1, \ldots, c_n) = L\,v - f$$

Ritz-Galerkinova metoda

$$(R,\, \varphi_k) = 0 \quad \longleftrightarrow \quad (L\,v,\, \varphi_k) = (f,\, \varphi_k), \qquad k = 1, \ldots, n$$

Metoda kolokacije

$$R(x_k; c_1, \ldots, c_n) = 0 \quad \longleftrightarrow \quad L\,v(x_k) = f(x_k), \qquad k = 1, \ldots, n$$

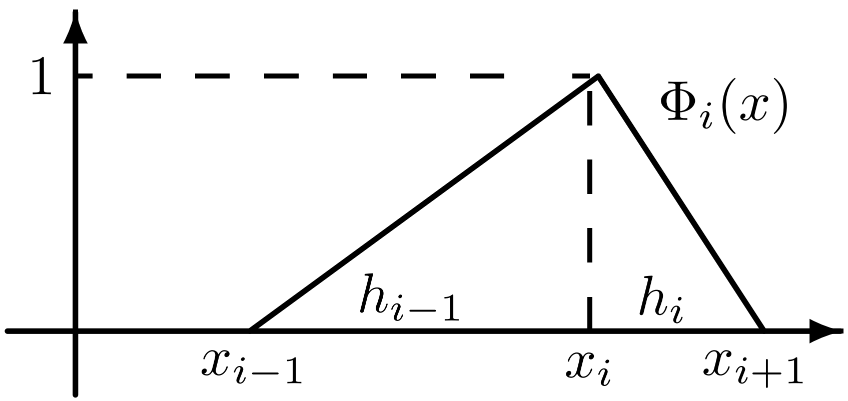
Metoda najmanjih kvadrata

$$\|R\|^2 = (R,\, R) = \min \quad \longleftrightarrow \quad (L\,v,\, L\phi_k) = (f,\, L\phi_k), \qquad k = 1, \ldots, n$$

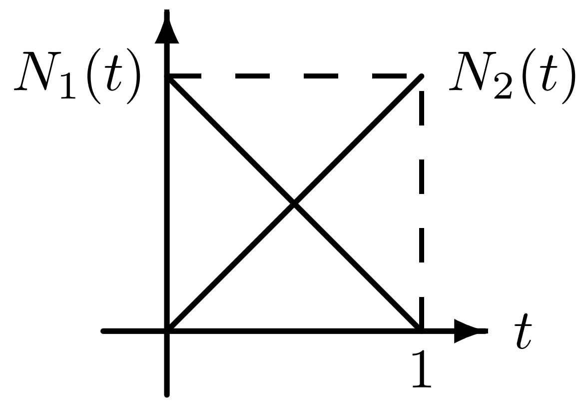
Metoda konačnog elementa – bazisne funkcije imaju kompaktan nosač

Primer: linearni B-splajn

$$\phi_i(x) = \begin{cases} 1 + \frac{x - x_i}{h_{i-1}}, & x \in [x_{i-1}, x_i] \\ 1 - \frac{x - x_i}{h_i}, & x \in [x_i, x_{i+1}] \\ 0, & x \notin [x_{i-1}, x_{i+1}] \end{cases}, \quad i = 0, \ldots, n,$$



krov funkcija



preslikavanje na kanonski element

$\diamond \qquad u''(x) - c\,u(x) = 0, \qquad u(0) = 1, \quad u'(1) = 0$

$\bar{\omega} = \{x_i \,|\, x_{i+1} - x_i = h_i, \sum_{i=0}^{n-1} h_i = 1\}$ $\qquad [x_i,\, x_{i+1}]$ konačni element

$$v(x) = \phi_0(x) + \sum_{i=1}^{n} v_i \phi_i(x), \qquad v(0) = \phi_0(0) = 1, \quad v_i = v(x_i), \quad i = 1, \ldots, n$$

$$(v'' - c\,v\,,\, \phi_j) = 0 \quad \longrightarrow \quad \int_0^1 v'(x)\phi_j'(x)\,dx + c\int_0^1 v(x)\phi_j(x)\,dx = 0$$

$$\int_0^1 (\phi_0'\phi_j' + c\,\phi_0\phi_j)\,dx + \sum_{i=1}^{n} v_i \int_0^1 (\phi_i'\phi_j' + c\,\phi_i\phi_j)\,dx = 0, \quad j = 1, \ldots, n$$

$\boxed{K\mathbf{v} = \mathbf{f}} \qquad k_{ij} = \int_0^1 (\phi_i'\phi_j' + c\phi_i\phi_j)\,dx, \quad f_j = -\int_0^1 (\phi_0'\phi_j' + c\phi_0\phi_j)\,dx$

$$k_{i,j} = 0, \quad |i-j| > 1, \qquad\qquad i = 1, \dots, n,$$

$$k_{i,i+1} = \int_{x_i}^{x_{i+1}} (\phi_i' \phi_{i+1}' + c\phi_i \phi_{i+1})\, dx, \qquad k_{i,i} = \sum_{j=0}^{1} \int_{x_{i-1+j}}^{x_{i+j}} ({\phi_i'}^2 + c{\phi_i}^2)\, dx$$

$$x \in [x_i, x_{i+1}] \qquad v(x) = v_i\, \phi_i(x) + v_{i+1}\, \phi_{i+1}(x) = v_i\, N_{1,i}(x) + v_{i+1}\, N_{2,i}(x)$$

$$\int_{x_i}^{x_{i+1}} \Big(\big(v_i N_{1,i}' + v_{i+1} N_{2,i}'\big) N_{j,i}' + c\big(v_i N_{1,i} + v_{i+1} N_{2,i}\big) N_{j,i} \Big)\, dx = 0,$$
$$j = 1, 2, \quad i = 1, \dots, n-1$$

$$x \in [x_0, x_1] \qquad v(x) = \phi_0(x) + v_1\, \phi_1(x) = N_{1,0}(x) + v_1\, N_{2,0}(x)$$

$$v_1 \int_{x_0}^{x_1} \big((N_{2,0}')^2 + c(N_{2,0})^2 \big)\, dx = - \int_{x_0}^{x_1} \big(N_{1,0}' N_{2,0}' + c\, N_{1,0} N_{2,0} \big)\, dx$$

$$x = x_i + h_i t = x_i(1-t) + x_{i+1} t, \quad t \in [0,1], \quad x \in [x_i,\, x_{i+1}],\, i = 1, \dots n-1$$

$$N_{1,i}(x) = \frac{x_{i+1} - x}{h_i}, \quad N_{2,i}(x) = \frac{x - x_i}{h_i} \quad \longrightarrow \quad N_1(t) = 1 - t, \quad N_2(t) = t$$

$$v(x(t)) = v_i N_1(t) + v_{i+1} N_2(t), \qquad x(t) = x_i N_1(t) + x_{i+1} N_2(t)$$

$$\int_0^1 \left(\frac{1}{h_i^2} \big(v_i N_1'(t) + v_{i+1} N_2'(t) \big) N_j'(t) + c \big(v_i N_1(t) + v_{i+1} N_2(t) \big) N_j(t) \right) h_i \, dt = 0$$

$$j = 1, 2$$

$$\int_0^1 \left(\frac{1}{h_i} (v_i - v_{i+1}) + c\, h_i \big(v_i (1-t) + v_{i+1} t \big)(1-t) \right) dt = 0$$

$$\int_0^1 \left(\frac{1}{h_i} (v_{i+1} - v_i) + c\, h_i \big(v_i (1-t) + v_{i+1} t \big) t \right) dt = 0$$

$$\frac{1}{h_0} v_1 + \frac{c\,h_0}{3} v_1 = \frac{1}{h_0} - \frac{c\,h_0}{6}$$

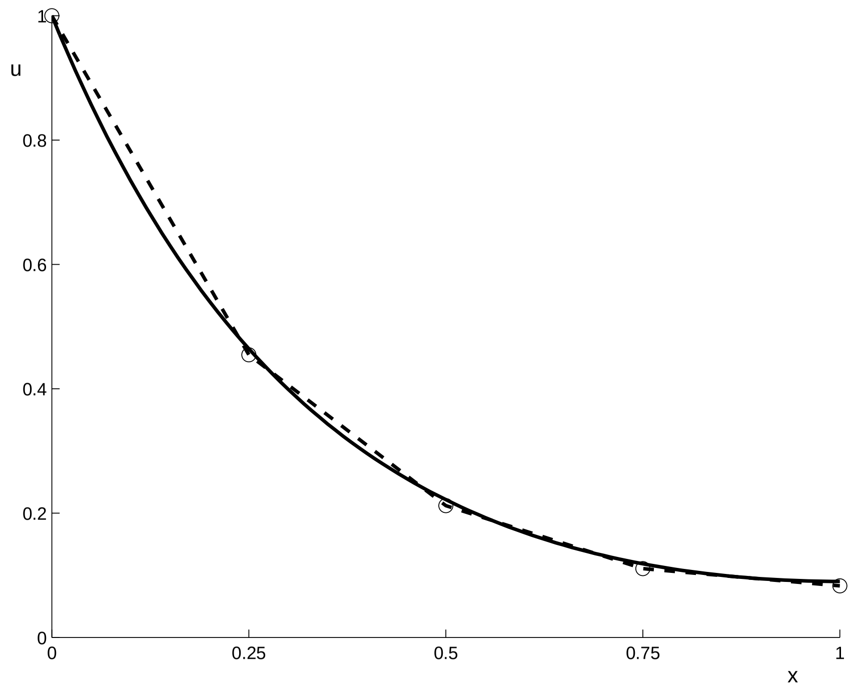
$$\left.\begin{aligned} \frac{v_i - v_{i+1}}{h_i} + \frac{c\,h_i}{6}\left(2v_i + v_{i+1}\right) = 0 \\ \frac{v_{i+1} - v_i}{h_i} + \frac{c\,h_i}{6}\left(v_i + 2v_{i+1}\right) = 0 \end{aligned}\right. \quad \longrightarrow \quad \begin{aligned} k_i\mathbf{v}^i \equiv (k_{s,i} + k_{m,i})\mathbf{v}^i = 0 \\ i = 1,\ldots,n-1 \end{aligned}$$

$$k_{s,i} = \frac{1}{h_i}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \qquad k_{m,i} = \frac{c\,h_i}{6}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \qquad \mathbf{v}^i = \begin{pmatrix} v_i \\ v_{i+1} \end{pmatrix}$$

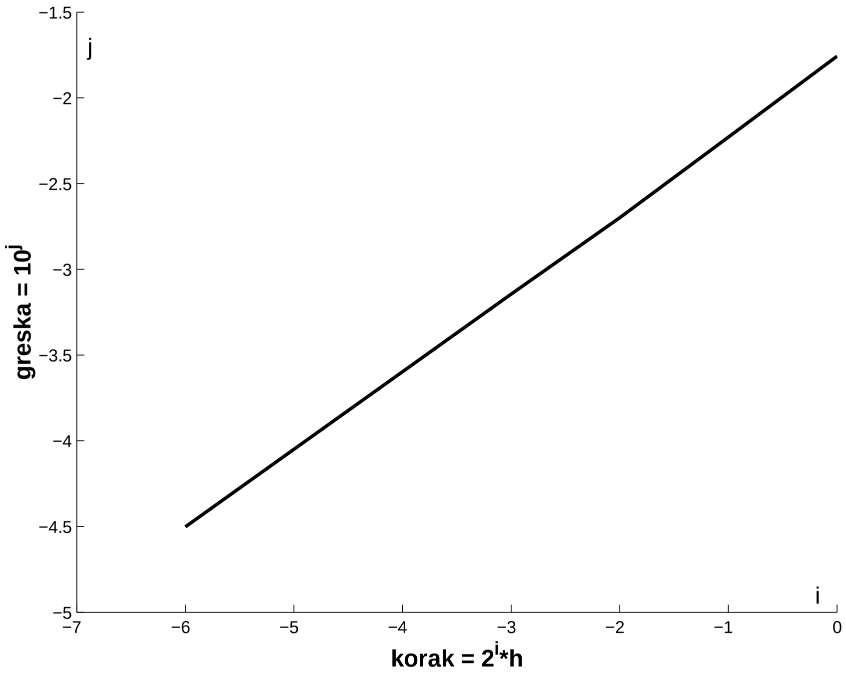
$$\boxed{(K_s + K_m)\,\mathbf{v} = \mathbf{f}}$$

$$K_s = \frac{1}{h}\begin{pmatrix} 2 & -1 & & & 0 \\ -1 & 2 & & & \\ & & \ddots & & \\ & & & 2 & -1 \\ 0 & & & -1 & 1 \end{pmatrix}, \quad K_m = \frac{ch}{6}\begin{pmatrix} 4 & 1 & & & 0 \\ 1 & 4 & & & \\ & & \ddots & & \\ & & & 4 & 1 \\ 0 & & & 1 & 2 \end{pmatrix}$$

Raspodela toplote određena metodom konačnog elementa



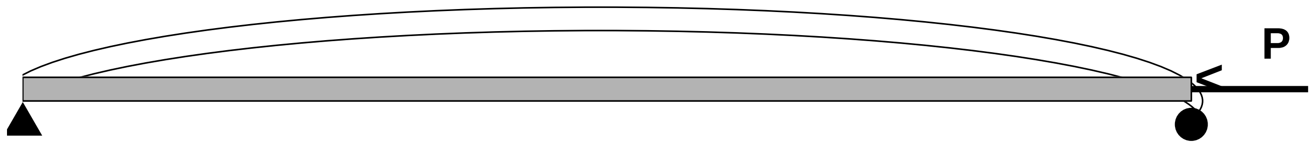
MKE sa 4 elementa



zavisnost greške od broja elemenata

◇ Deformacija grede pod uticajem bočne sile

$$u''(x) + c\,u(x) = 0$$
$$u(0) = u(1) = 0$$

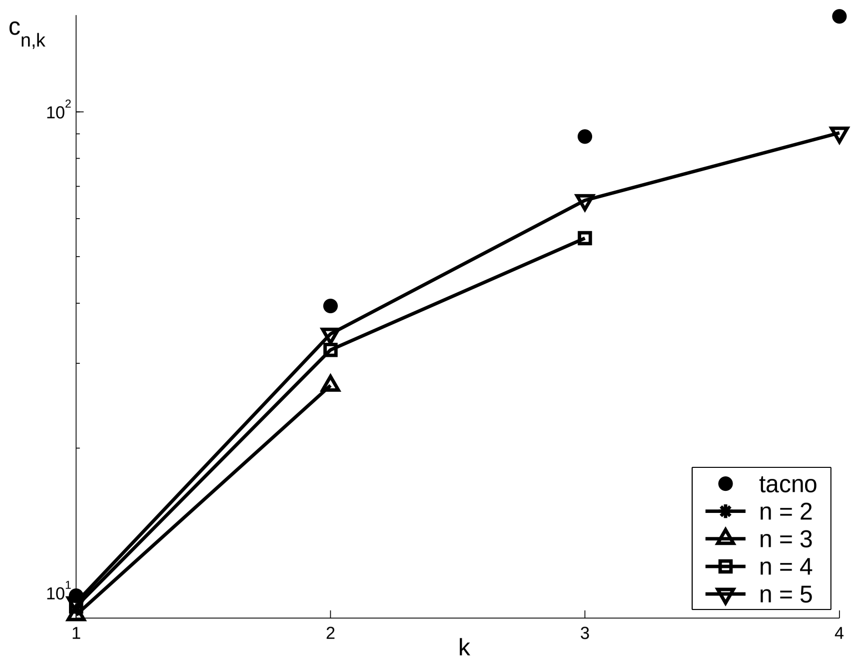


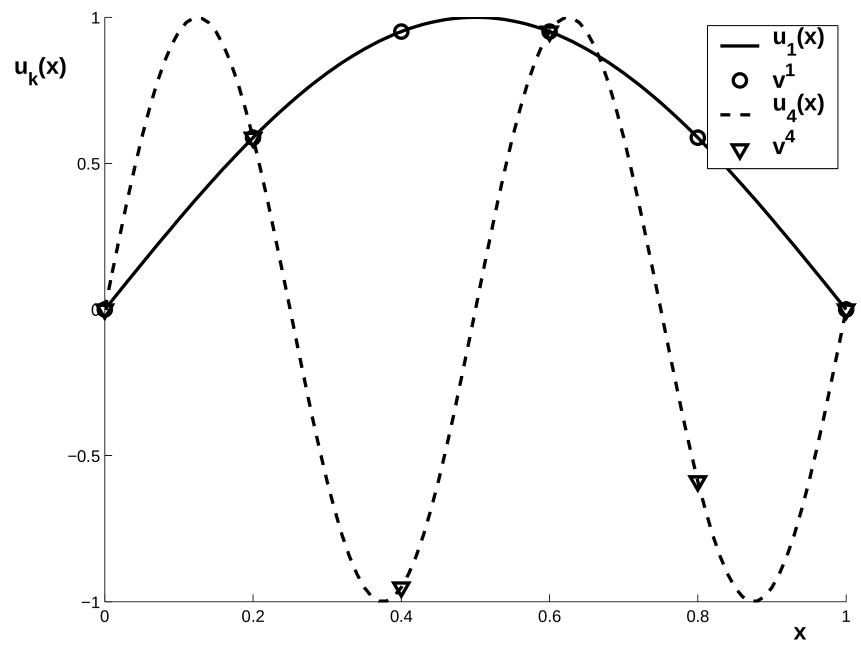
$$u_k(x) = \sin k\pi x, \qquad c_k = k^2\pi^2, \qquad k = 1, 2, \ldots$$

$$v_{\bar{x}x,i} + c_n\, v_i = 0, \quad i = 1, \ldots, n-1, \qquad v_0 = v_n = 0$$

$$A_n \mathbf{v}_n = c_n \mathbf{v}_n, \qquad A_n = \frac{1}{h^2} \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & \\ & & \ddots & \\ 0 & & & 2 \end{pmatrix}$$

$$c_{n,k} = \frac{4}{h^2}\sin^2\frac{k\pi h}{2}, \qquad \mathbf{v}_{n,k} = \begin{pmatrix} \sin k\pi x_1 \\ \vdots \\ \sin k\pi x_{n-1} \end{pmatrix}, \qquad k = 1,\ldots,n-1$$





Aproksimacije sopstvenih vrednosti, sopstvenih funkcija

Mešoviti problemi

◇ Raspodela toplote sa vremenom u tankom štapu, koji je izolovan na oba kraja (granični uslovi) i ima zadatu raspodelu temperature na početku posmatranja procesa (početni uslov)

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}, \qquad x \in [-1, 1], \quad t > 0$$

granični uslovi $$T(-1, t) = T(1, t) = 0, \quad t > 0,$$

početni uslov $$T(x, 0) = e^{-10x^2}, \quad x \in [-1, 1]$$

Jednačina provođenja toplote je jednačina paraboličkog tipa (difuzija, rasipanje)

$$\frac{\partial u}{\partial t} = \Delta u + q, \qquad \Delta u \equiv \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_m^2}$$

Diferencijska šema

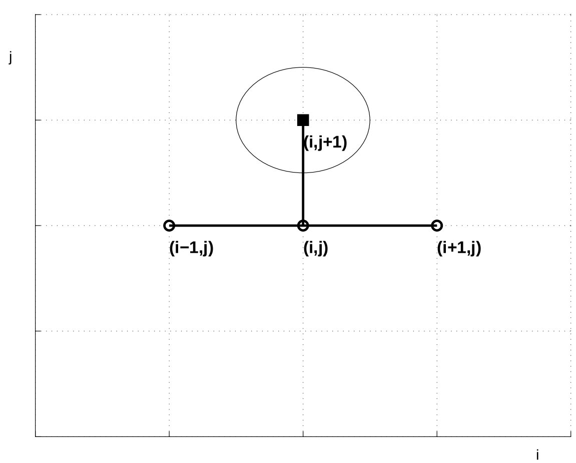
$$\omega_h = \{x_i, \,\big|\, x_i = ih,\ i = -n, \ldots, n,\ h = \frac{1}{n}\}$$

$$\omega_\tau = \{t_j, \,\big|\, t_j = j\tau,\ j = 0, \ldots, m,\ \tau = \frac{tmax}{m}\}$$

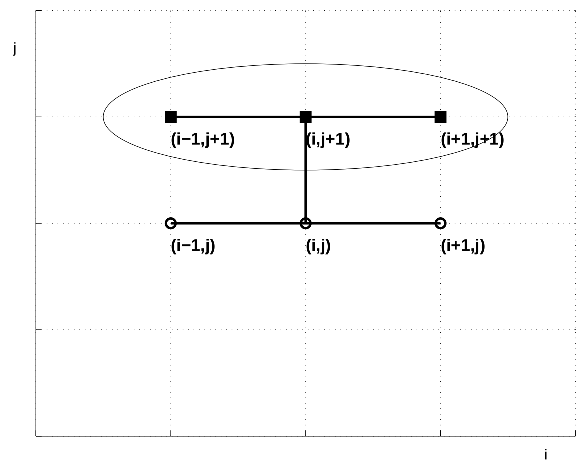
$$\omega_h \times \omega_\tau = \{(x_i, t_j),\ i = -n, \ldots, n,\ j = 0, \ldots, m\}$$

$$v_{t,i}^{j} = (1-\sigma)\, v_{\overline{x}x,i}^{j} + \sigma\, v_{\overline{x}x,i}^{j+1} \qquad i = -n+1, \ldots, n-1,$$

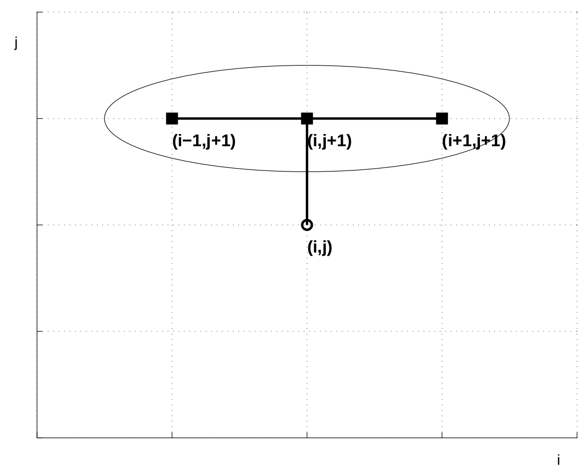
$$v_{-n}^{j} = v_n^{j} = 0, \qquad v_i^0 = e^{-10x_i^2} \qquad j = 0, \ldots, m-1,$$



eksplicitna ($\sigma = 0$)



Crank-Nicolson-ova ($\sigma = 1/2$)

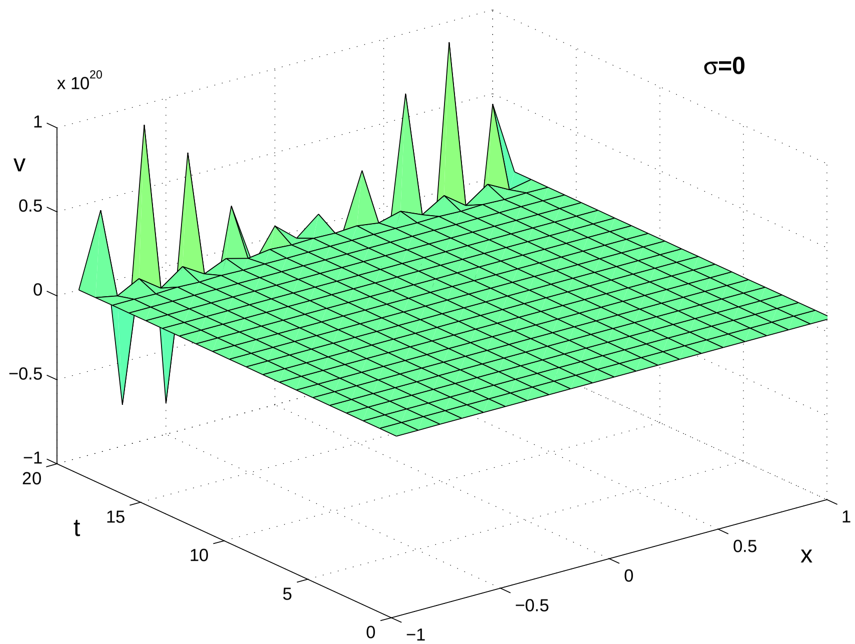


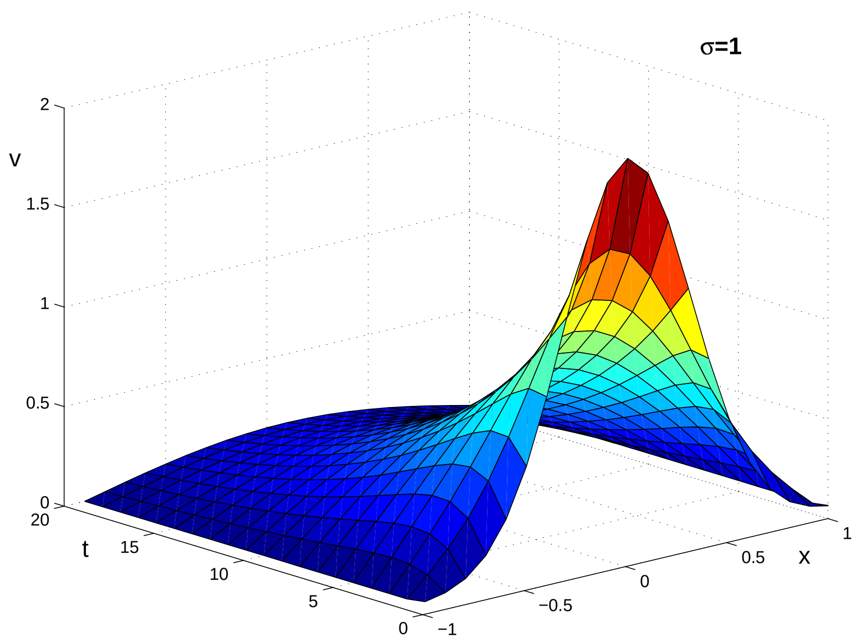
implicitna ($\sigma = 1$)

$$-\frac{\sigma\tau}{h^2}v_{i-1}^{j+1} + \left(1 + 2\frac{\sigma\tau}{h^2}\right)v_i^{j+1} - \frac{\sigma\tau}{h^2}v_{i+1}^{j+1} \qquad i = -n+1, \ldots, n-1,$$

$$= \frac{(1-\sigma)\tau}{h^2}v_{i-1}^{j} + \left(1 - \frac{2(1-\sigma)\tau}{h^2}\right)v_i^{j} + \frac{(1-\sigma)\tau}{h^2}v_{i+1}^{j}$$

$$v_{-n}^j = v_n^j = 0$$

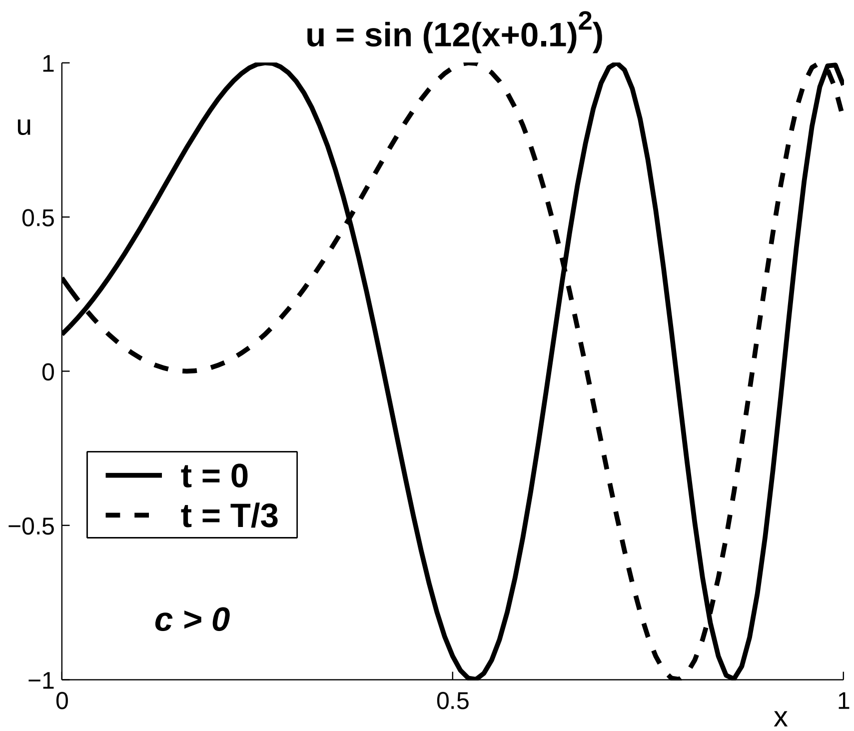




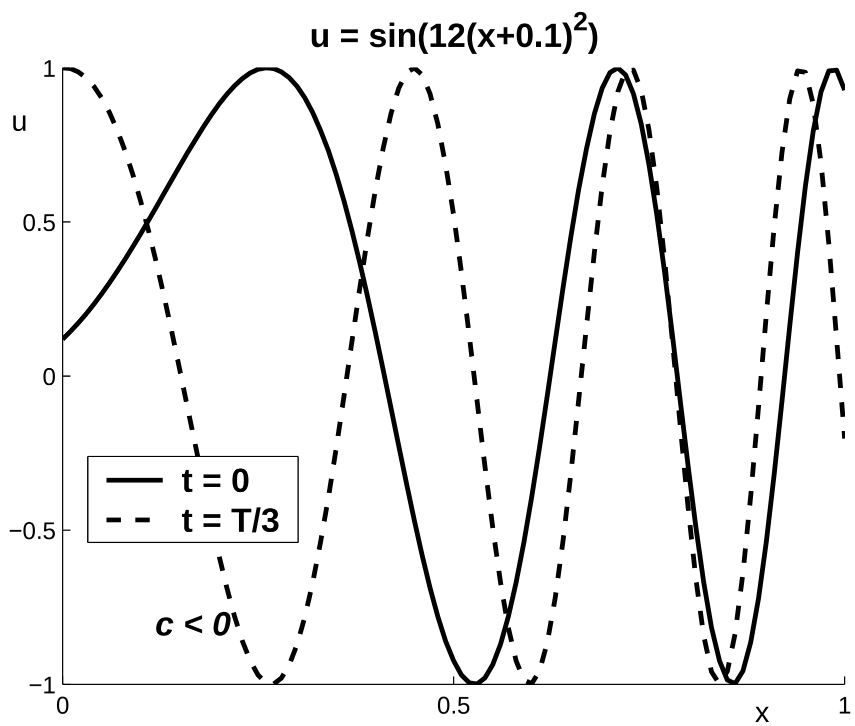
$h = 0.1$, $\tau = 0.05$: eksplicitna šema implicitna šema

◇ Strujanje (konvekcija) konstantnom brzinom $|c|$

$$\frac{\partial u}{\partial t} + c\,\frac{\partial u}{\partial x} = 0, \quad u(x,0) = u_0(x) \qquad \longrightarrow \qquad u(x,t) = u_0(x - ct)$$



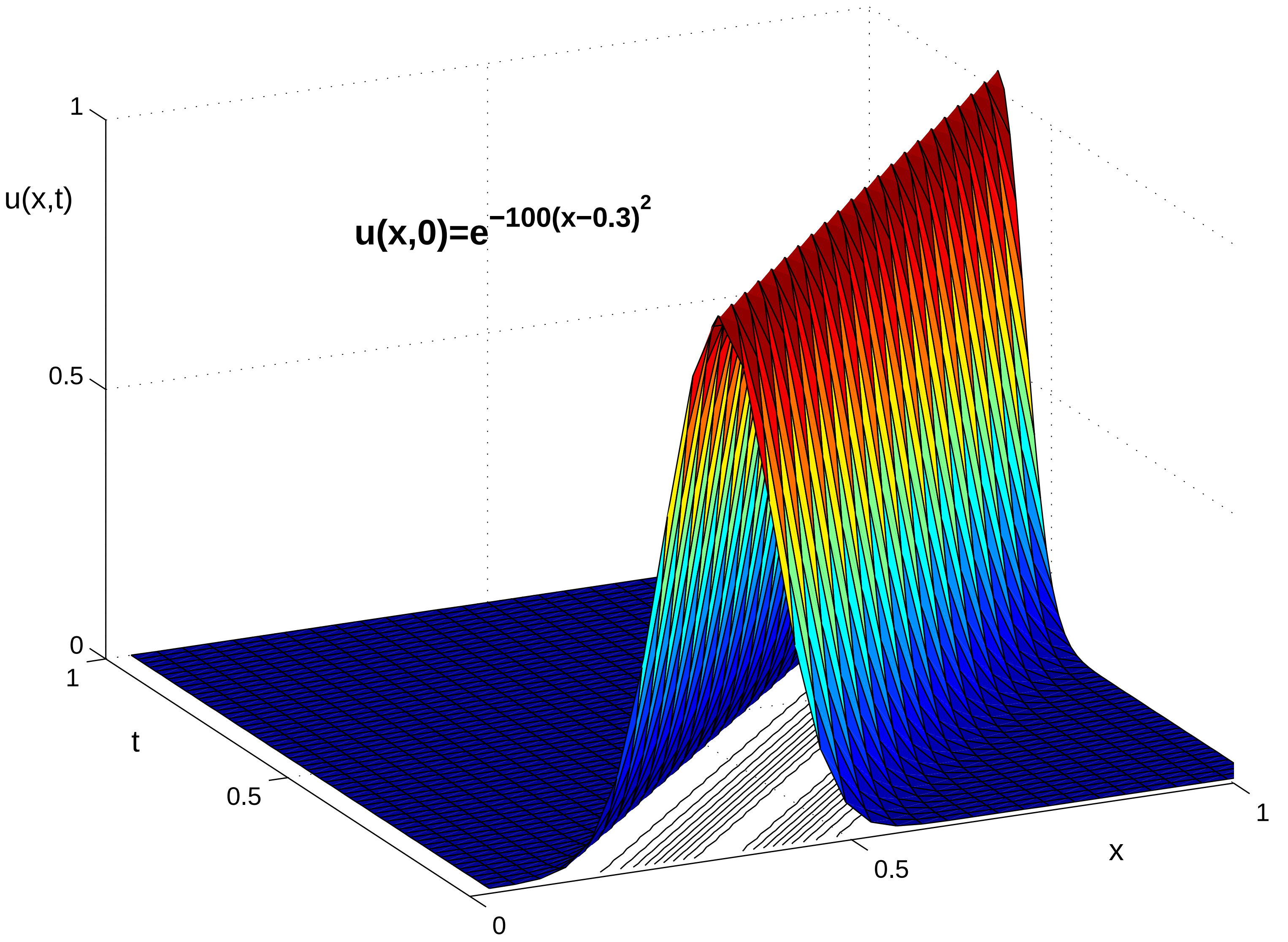
u desno ($c > 0$)



u levo ($c < 0$)

MatLab: wave1.m

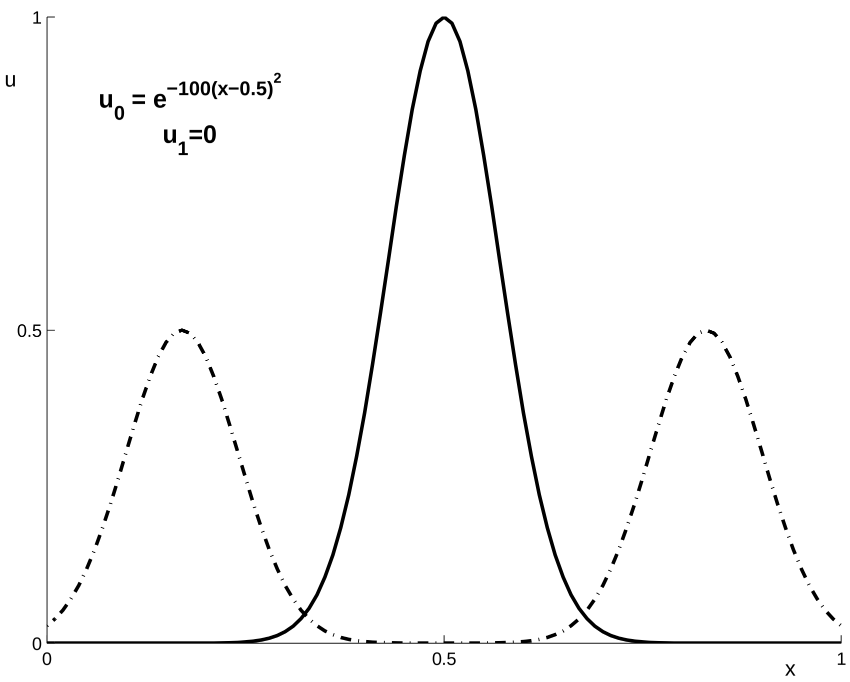
Kretanje duž karakteristika: $u(x,t) = u_0(X)$ za $x - ct = X$



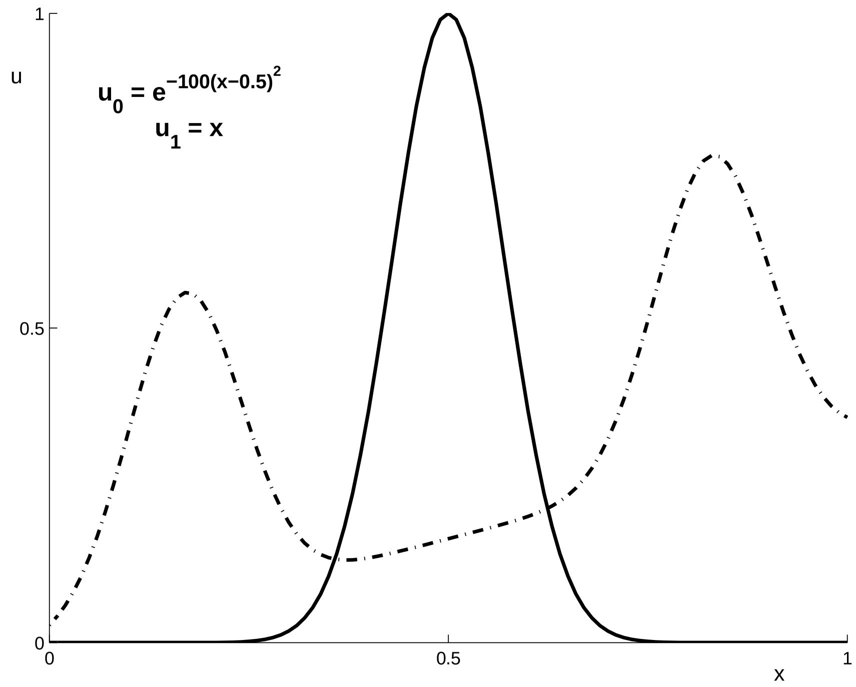
◇ Talasna jednačina – Cauchy-jev problem

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2}, \qquad x \in \mathcal{R}, \quad t > 0,$$

$$u(x,0) = u_0(x), \qquad \frac{\partial u(x,0)}{\partial t} = u_1(x)$$

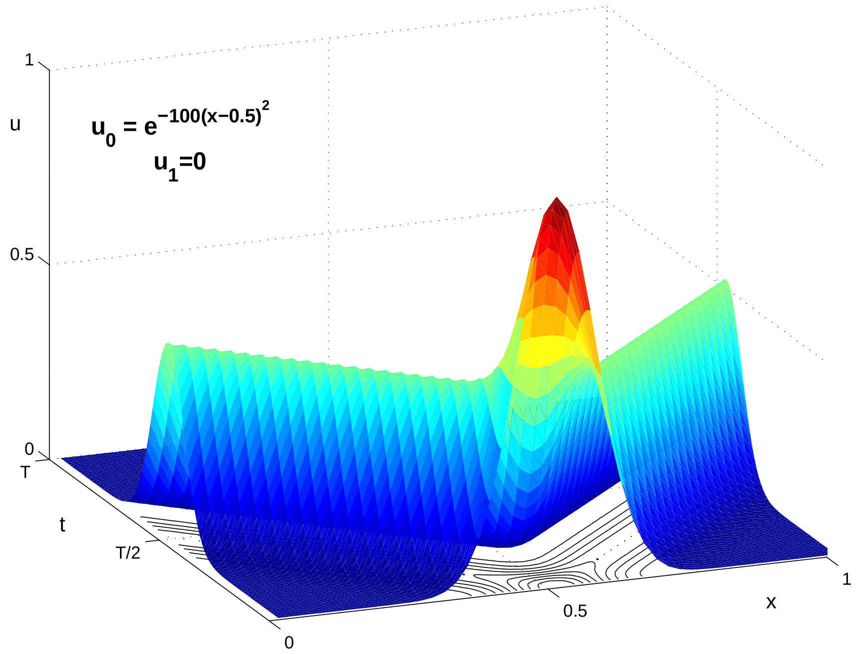


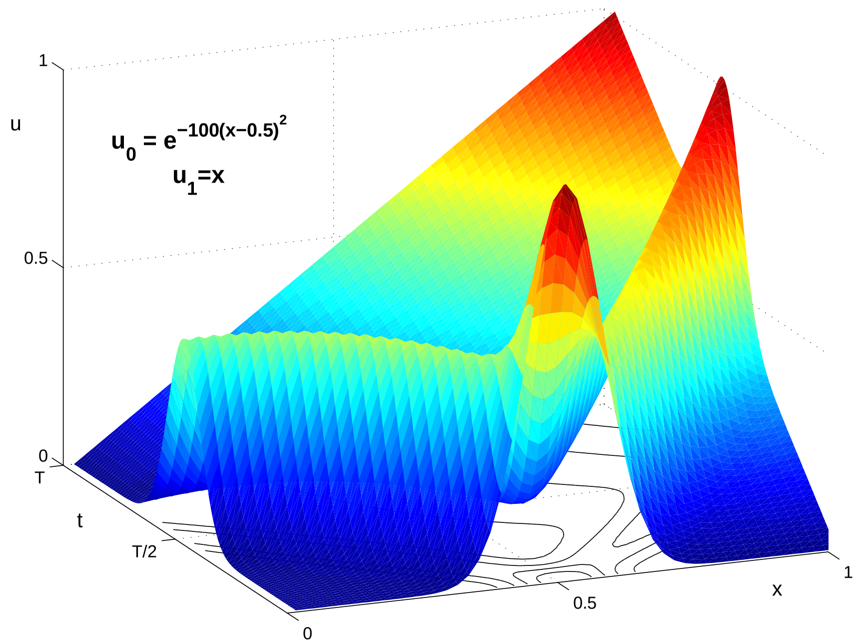
$u_1 = 0$



$u_1 = x$

Prethodna slika u tri dimenzije



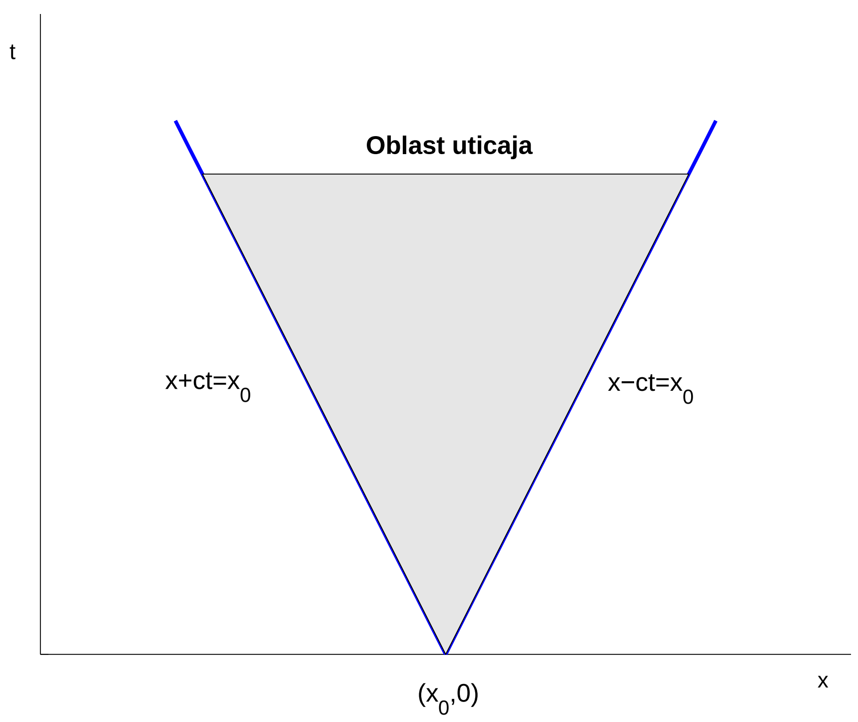


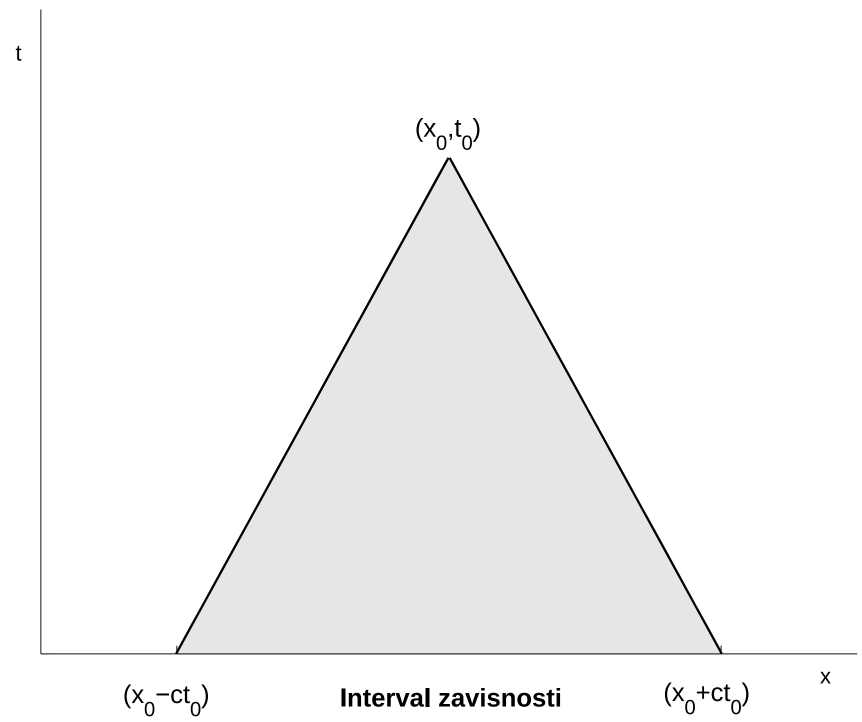
Talasna jednačina je jednačina hiperboličkog tipa (konvekcija, strujanje)

$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u + q, \qquad \Delta u \equiv \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_m^2}$$

$$u(x,t) = \frac{1}{2}\big(u_0(x-ct) + u_0(x+ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} u_1(s)\,ds$$

Oblasti zavisnosti i uticaja su određene karakteristikama

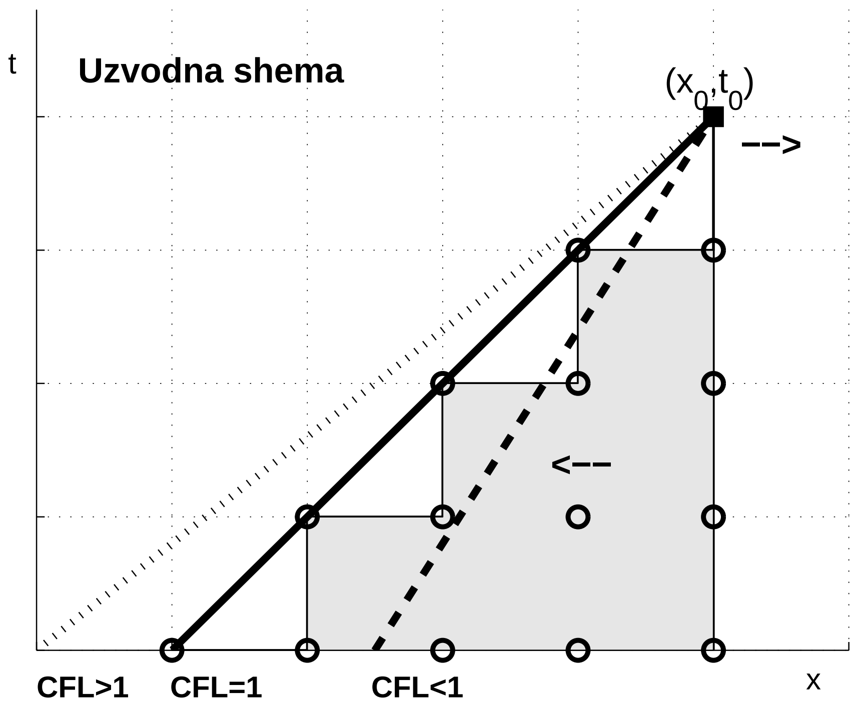


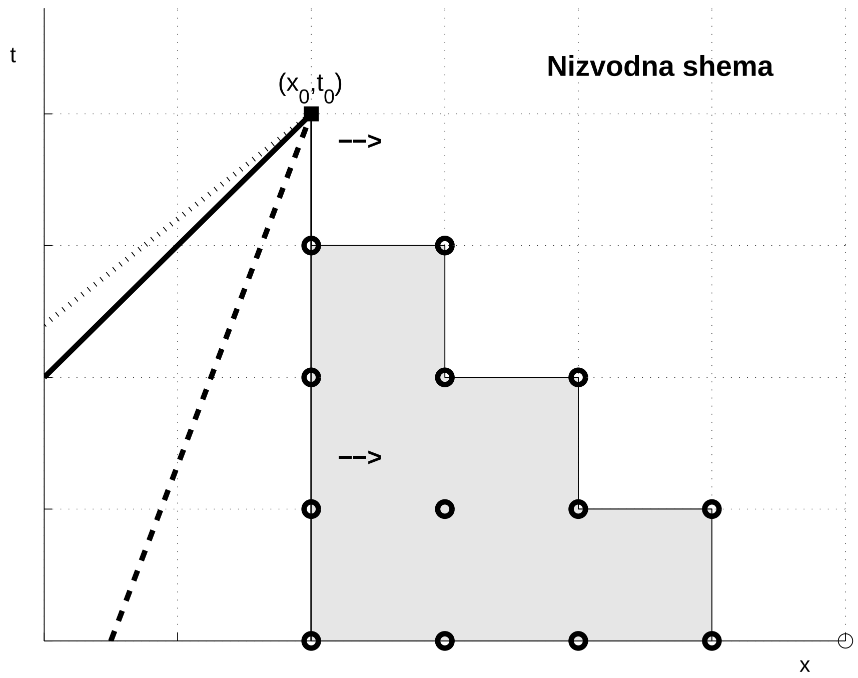


$$\frac{\partial u}{\partial t} + c\,\frac{\partial u}{\partial x} = 0 \qquad \frac{1}{\tau}\left(v_i^{j+1} - v_i^j\right) + c\frac{1}{h}\left(v_i^j - v_{i-1}^j\right) = 0$$

$$(c > 0) \qquad\qquad \longrightarrow$$

$$\frac{\partial u}{\partial t} - c\,\frac{\partial u}{\partial x} = 0 \qquad \frac{1}{\tau}\left(v_i^{j+1} - v_i^j\right) - c\frac{1}{h}\left(v_{i+1}^j - v_i^j\right) = 0$$

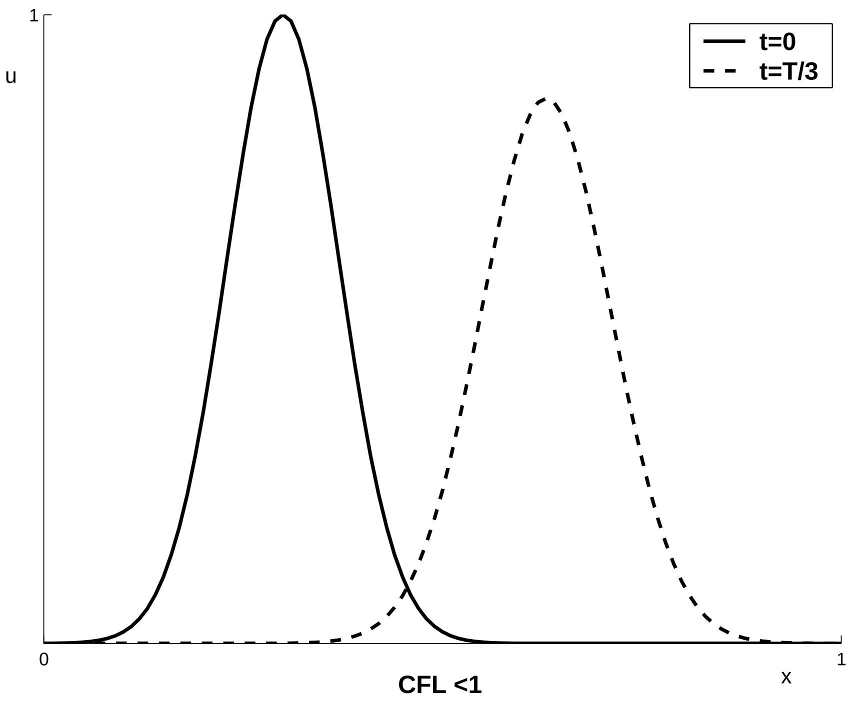




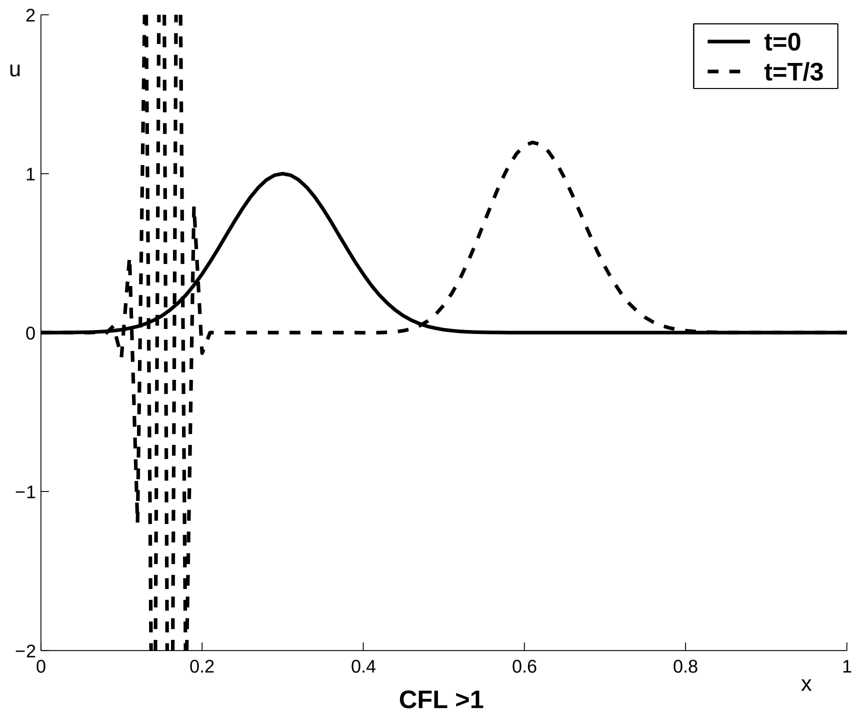
Optimalan izbor šeme: DA NE

Uticaj odnosa koraka po vremenu i prostoru na stabilnost uzvodne šeme

$$\text{CFL} = c\,\frac{\tau}{h} \begin{cases} \le 1, & \text{stabilna} \\ > 1, & \text{nestabilna} \end{cases} \qquad (c > 0)$$

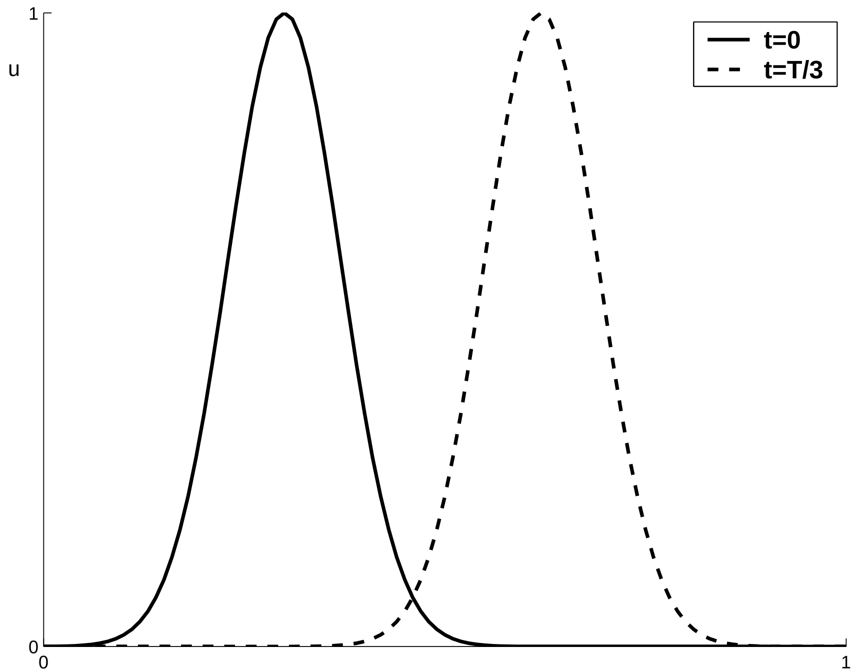


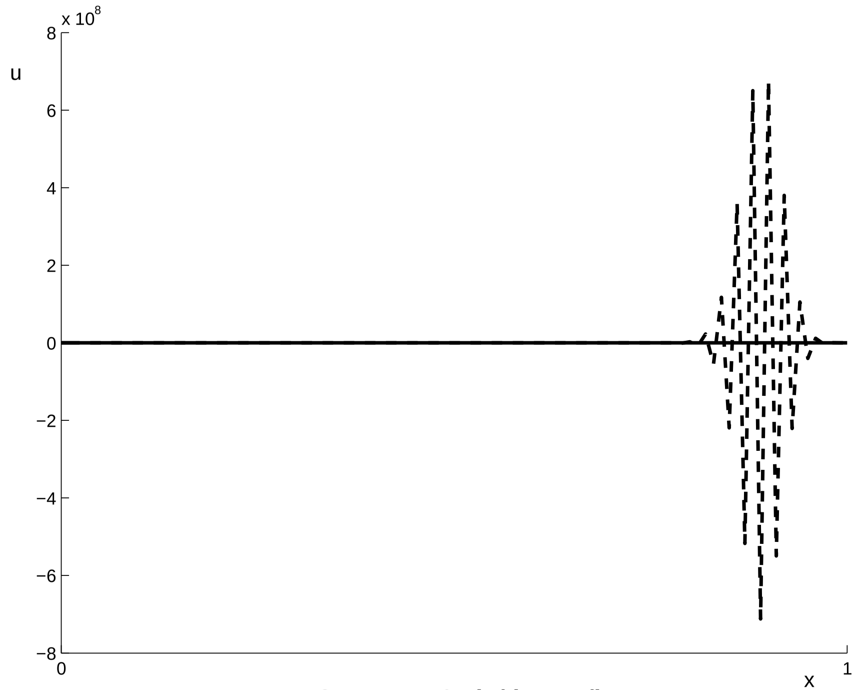
$CFL < 1$, disipacija



$CFL > 1$, nestabilnost

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0, \qquad u_0(x) = e^{-100(x-0.3)^2}, \qquad x \in \mathcal{R}$$





$CFL = 1$, uzvodna šema
- bez deformacije

$CFL = 1$, nizvodna šema
- nestabilnost

◇ Matematički model žice koja osciluje (mešoviti problem), uz pretpostavke

- učvršćena je na krajevima (dva granična uslova)
- izložena je dejstvu sile $f(x,t)$ (slobodni član jednačine)
- funkcija $u_0(x)$ opisuje oblik žice na početku posmatranja (prvi početni uslov)
- funkcija $u_1(x)$ opisuje brzinu oscilovanja žice na početku posmatranja (drugi početni uslov)

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + x^2 + t^2, \qquad 0 < x < 1, \quad t > 0,$$

$$u(0,t) = u(1,t) = 0,$$

$$u(x,0) = u_0(x) \equiv \sin(\pi x), \qquad \frac{\partial u}{\partial t}(x,0) = u_1(x) \equiv 0$$

Za dobijanje aproksimacije tačnosti $O(h^2 + \tau^2)$ koristimo razvoj

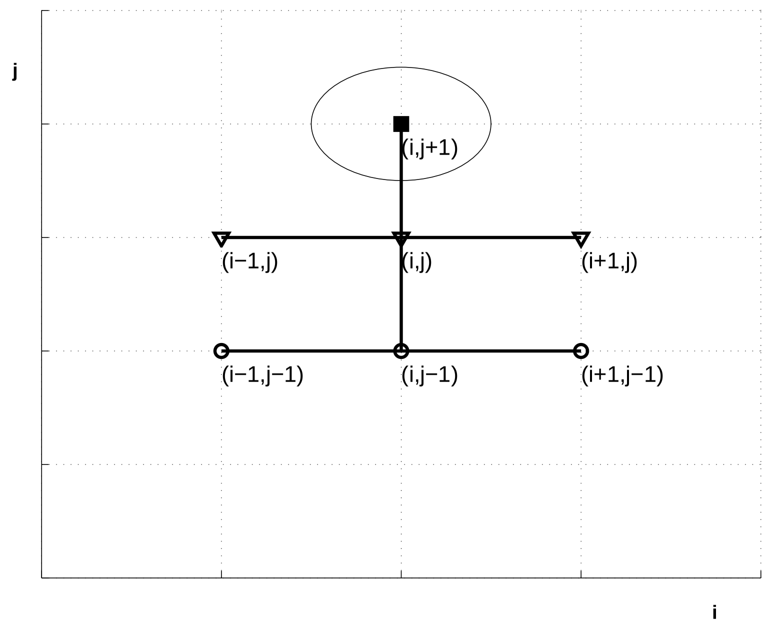
$$u_t(x,0) = \frac{1}{\tau}\big(u(x,\tau) - u(x,0)\big) = \frac{\partial u}{\partial t}(x,0) + \frac{\tau}{2}\frac{\partial^2 u}{\partial t^2}(x,0) + O(\tau^2)$$

Diferencijska šema

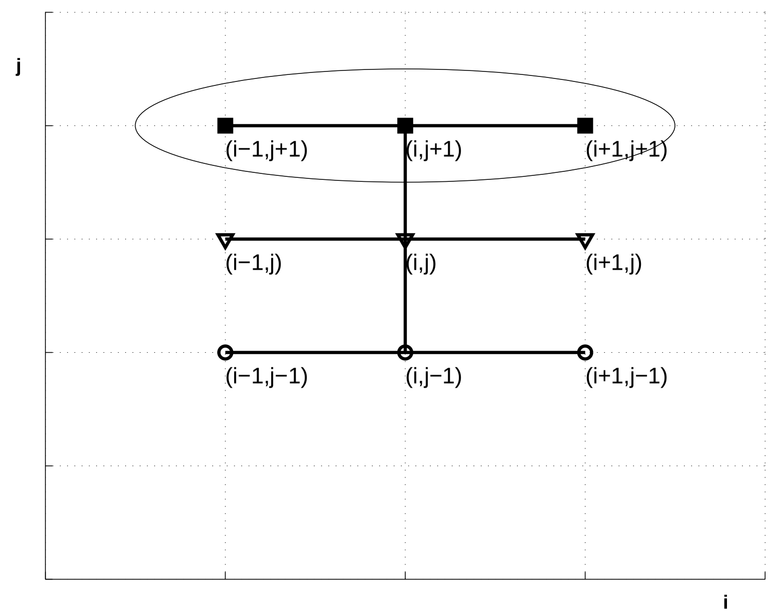
$$v_{\bar{t}t,i}^{j} = \sigma_1\, v_{\bar{x}x,i}^{j+1} + (1-\sigma_1-\sigma_2)\, v_{\bar{x}x,i}^{j} + \sigma_2\, v_{\bar{x}x,i}^{j-1} + x_i^2 + t_j^2,$$

$$v_0^j = v_n^j = 0 \qquad i = 1,\ldots,n-1, \quad j = 1,\ldots,m-1,$$

$$v_i^0 = \sin(\pi x_i), \quad v_i^1 = \sin(\pi x_i) + \frac{\tau^2}{2}\left(x_i^2 - \pi^2 \sin(\pi x_i)\right)$$



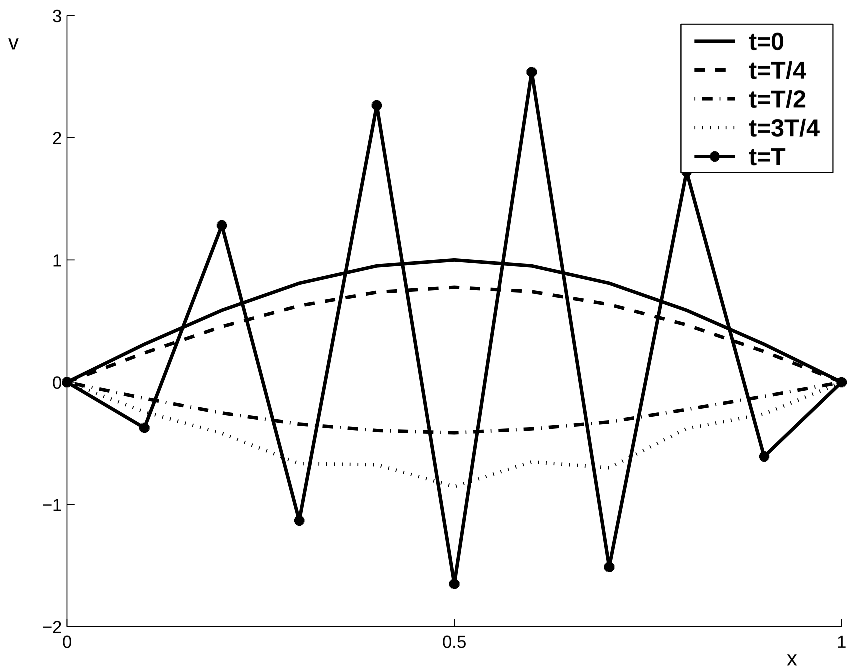
eksplicitna $(\sigma_1 = 0)$



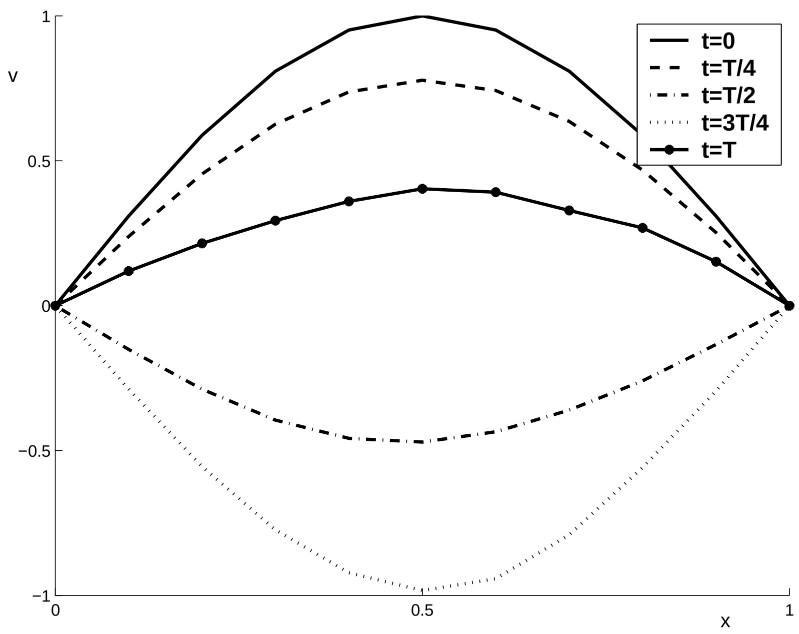
implicitna $(\sigma_1 \neq 0)$

Grafici prikazuju položaj žice koja osciluje u različitim vremenskim trenucima

Određeni su diferencijskom šemom za $CFL = \frac{\tau}{h} = 1.1$



eksplicitna šema $\sigma_1 = \sigma_2 = 0$



implicitna šema $\sigma_1 = 1/3, \sigma_2 = 2/3$