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REFLEXIVE CACTI: A SURVEY

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Dedicated to Professor Zoran Radosavljević

A graph is called reflexive if its second largest eigenvalue does not exceed 2. We survey the results on reflexive cacti obtained in the last two decades. We also discuss various patterns of appearing of Smith graphs as subgraphs of reflexive cacti. In the Appendix, we survey the recent results concerning reflexive bipartite regular graphs.

1. INTRODUCTION

Throughout the paper, we consider simple graphs. If G is such a graph with n vertices and adjacency matrix A, then its characteristic polynomial P_G is just the characteristic polynomial of A, that is $P_G(\lambda) = \det(\lambda I - A)$. Its eigenvalues are real and they are denoted $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. The collection of eigenvalues (with repetitions) forms the spectrum of G. The largest eigenvalue is called the index of G. If G is connected, then its index is a simple eigenvalue.

The second largest eigenvalue of a graph G is related to structural invariants such as diameter and connectivity [**1**, **31**], and graphs with comparatively small value for λ_2 attracted the most of attention. Several bounds for λ_2 have been investigated in more details: different classes of graphs with the property $\lambda_2 \leq r$ have been described for $r \in \left\{\frac{1}{3}, \sqrt{2}-1, \frac{\sqrt{5}-1}{2}, 1, \sqrt{2}, \frac{\sqrt{5}+1}{2}, \sqrt{3}, 2\right\}$. As r grows the research becomes more complex. All graphs with $\lambda_2 \leq \frac{1}{3}$ are determined in one theorem [**2**]. Graphs with $\lambda_2 \leq \sqrt{2}-1$ are also completely determined [**18**].

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Those with $\lambda_2 \leq \frac{\sqrt{5}-1}{2}$ are studied in several papers, for example [**4**, **5**, **6**, **29**, **35**], but they are not completely determined. Graphs with $\lambda_2 \leq 1$ are investigated in even more papers and they are completely described within certain classes (see for example [**3**, **7**, **11**, **12**, **32-34**, **36**]). For graphs with $\lambda_2 \leq \sqrt{2}$ and $\lambda_2 \leq \sqrt{3}$ the reader may see [**8**, **11**, **14**, **35**], and for those with $\lambda_2 \leq \frac{\sqrt{5}+1}{2}$ see [**13**].

By a (proper) subgraph of graph G we consider an induced subgraph of G; if H is a (proper) subgraph of G, we say that G is a (proper) supergraph of H. A cactus is a connected graph such that any two cycles induced in it have at most one common vertex. If all cycles of a cactus have a unique common vertex, we say that they form a (single) bundle. The remaining terminology and notation are taken from [**4**].

Our subject are graphs with the property $\lambda_2 \leq 2$. Such graphs are commonly known as reflexive since they represent the Lorentzian counterparts of the spherical and Euclidean graphs which occur in the theory of reflection groups [**17**]. Of course, since the spectrum of a disconnected graph is the union of the spectra of its components, it is natural for such an investigation to treat only connected graphs.

The first brief survey on reflexive graphs can be found in [**19**]. The investigations on these graphs are mostly focused on cacti whose cycles do not form a bundle, since it occurs that classes of multicyclic cacti whose cycles do form a bundle are large and complex. Note that all reflexive trees are determined in [**16**].

Which bicyclic graphs are reflexive? This question, posed by Radosavljević and Simić in [**25**], was the starting point of the research on reflexive cacti. In that article, besides determining all maximal reflexive bicyclic cacti with the bridge between its cycles, the authors gave the RS-Theorem that proved to be essential in subsequent investigations.

After that, a sequence of papers authored by Radosavljević, Rašajski, Mihailović, and Koledin appeared. The main subject of all of them was determination of reflexive cacti with prescribed properties and developing spectral tools for their characterizations. An important result was obtained by Radosavljević and Rašajski [**24**] who proved that any reflexive cactus whose cycles do not form a bundle and to which RS-Theorem cannot be applied has at most five cycles.

Starting from these two conditions further investigations were mostly focused on the cyclic structure and determination of reflexive cacti in particular cases. So far, all reflexive cacti with five and four cycles, all tricyclic reflexive cacti (except for one class, which has been partially determined) and some classes of bicyclic and unicyclic cacti have been described. It is worth mentioning that the doctoral theses of Rašajski [**26**] and Mihailović [**13**] and the master theses of Mihailović and Koledin are mostly based on these investigations (for all of them the mentor was professor Zoran Radosavljević).

The purpose of this paper is to survey mentioned results. Our objective is to describe main ideas and the cyclic structure, while for more details the reader can consult the corresponding references.

In Section 2 we give some preparatory results. Main considerations are given in Sections 3 and 4. A short conclusion is separated into Section 5. In the Appendix, we give recent results concerning reflexive bipartite regular graphs.

2. PRELIMINARIES

We start with a well-known result.

Interlacing Theorem. *Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of a simple graph G and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m$ the eigenvalues of its subgraph H. Then the inequalities $\lambda_{n-m+i} \leq \mu_i \leq \lambda_i$ $(i = 1, 2, \ldots, m)$ hold.*

By this theorem, the property $\lambda_2 \leq 2$ (and generally $\lambda_i \leq r$) is a hereditary one, that is if the second largest eigenvalue of a graph G does not exceed 2, then the same holds for any of its subgraphs. That is why such investigations are directed either to determining all maximal graphs or to finding all minimal forbidden graphs (inside the observed class) for this property.

In order to decide if a given graph G is reflexive or not, we usually use the following conclusion: If $\lambda_3(G) < 2 < \lambda_1(G)$, then $\mathrm{sgn} P_G(2) = \mathrm{sgn}(\lambda_2(G) - 2)$. Therefore, if we prove $\lambda_3(G) < 2 < \lambda_1(G)$ (for example, by Interlacing Theorem), it remains to compute the value $P_G(2)$, for which we use next lemmas and corollaries.

Lemma 1. [**28**] (Schwenk) *If $G = G_1 \cdot G_2$ where G_1 and G_2 are two rooted graphs with the roots x_1 and x_2, respectively, and a coalescence is formed in roots, then for the characteristic polynomial of G it holds that*

$$P_G(\lambda) = P_{G_1}(\lambda)P_{G_2-x_2}(\lambda) + P_{G_1-x_1}(\lambda)P_{G_2}(\lambda) - \lambda P_{G_1-x_1}(\lambda)P_{G_2-x_2}(\lambda).$$

Lemma 2. [**28**] (Schwenk) *Given a graph G, let $C(v)$ and $C(uv)$ denote the set of all cycles containing a vertex v and an edge uv of G, respectively. Then,*

$$1)\ P_G(\lambda) = \lambda P_{G-v}(\lambda) - \sum_{u\in \mathrm{Adj}(v)} P_{G-v-u}(\lambda) - 2\sum_{C\in \mathrm{C}(v)} P_{G-V(C)}(\lambda),$$

$$2)\ P_G(\lambda) = P_{G-uv}(\lambda) - P_{G-v-u}(\lambda) - 2\sum_{C\in \mathrm{C}(uv)} P_{G-V(C)}(\lambda),$$

where $\mathrm{Adj}(v)$ *denotes the set of neighbors of v, while $G - V(C)$ is the graph obtained from G by removing the vertices belonging to the cycle C.*

For the next two useful corollaries see, for example, [**4**].

Corollary 1. [**4**] *Let G be a graph obtained by joining a vertex v_1 of a graph G_1 to a vertex v_2 of a graph G_2 by an edge. Let G_1' (G_2') be the subgraph of G_1 (G_2) obtained by deleting a vertex v_1 (v_2) from G_1 (resp. G_2). Then*

$$P_G(\lambda) = P_{G_1}(\lambda)P_{G_2}(\lambda) - P_{G_1'}(\lambda)P_{G_2'}(\lambda).$$

Corollary 2. [**4**] *Let G be a graph with a pendent edge v_1v_2, v_1 being of degree 1. Then*

$$P_G(\lambda) = \lambda P_{G_1}(\lambda) - P_{G_2}(\lambda),$$

where G_1 (G_2) *is obtained from the graph* G (*resp.* G_1) *by deleting the vertex* v_1 (*resp.* v_2).

Radosavljević and Simić [**25**] have shown the following result. A graph is called positive, null or negative depending on whether its index is greater than, equal to or less than 2.

RS-Theorem. [**25**] *Let* G *be a graph with a cut-vertex* u.

1) *If at least two components of* $G-u$ *are positive or if only one is positive and some of the rest are null, then* $\lambda_2(G) > 2$.

2) *If at least two components of* $G-u$ *are null and any other non-positive, then* $\lambda_2(G) = 2$.

3) *If at most one component of* $G-u$ *is null and the rest are negative, then* $\lambda_2(G) < 2$.

For the described case this theorem is stronger than the Interlacing Theorem.

For some graphs with a cut-vertex, RS-Theorem gives an answer whether they are reflexive or not. We call such graphs RS-decidable. Otherwise, they are said to be RS-undecidable. The theorem itself is generalized for any positive real number instead of 2 [**14**].

Obviously, the number of cycles in an RS-decidable reflexive graph is not limited.

3. REFLEXIVE CACTI WITHOUT THE BUNDLE OF ALL CYCLES

3.1. Smith graphs

Bearing in mind the essential part they play in description and characterization of reflexive graphs, let us start this survey by presenting Smith graphs [**30**], even though their presence and role will be discussed thoroughly in the next section.

Smith graphs are the only connected graphs for which $\lambda_1 = 2$ holds and they are depicted in Fig. 1.

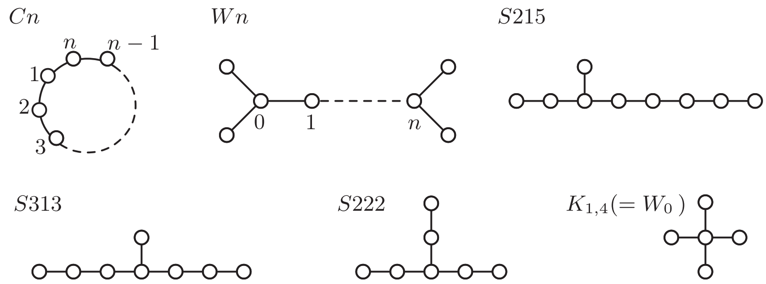

Figure 1.

At first, it was noticed that if after removal of a cut-vertex of a given graph we get two Smith graphs, for example as in Fig. 2, then, applying the Interlacing Theorem, we get that $\lambda_2 = 2$ holds for the initial graph.

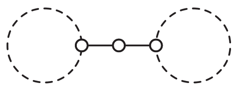

Figure 2.

This led to a more general idea that resulted in a variant of the RS-Theorem expressed in terms of Smith graphs, which appeared as very important in the investigations that followed. If after removal of a cut-vertex we get an arbitrary number of components, this theorem gives the answer about the reflexiveness of the original graph depending on these components. Notice that each component is either a Smith graph, a proper subgraph of a Smith graph or a proper supergraph of a Smith graph.

RS-Theorem. [**25**, see also **27**] *Let G be a graph with a cut-vertex u.*

1) *If at least two components of $G-u$ are supergraphs of Smith graphs and if at least one of them is a proper supergraph, then $\lambda_2(G) > 2$.*

2) *If at least two components of $G-u$ are Smith graphs and the rest are subgraphs of Smith graphs, then $\lambda_2(G) = 2$.*

3) *If at most one component of $G-u$ is a Smith graph and the rest are proper subgraphs of Smith graphs, then $\lambda_2(G) < 2$.*

The theorem does not answer the question whether the graph is reflexive if after removing the cut-vertex we get exactly one proper supergraph, while all others are proper subgraphs of Smith graphs. Therefore, the focus in further investigations is always on them.

Recently, the maximum number of cycles in an RS-undecidable reflexive cactus whose cycles do form a bundle has been determined [**13**] and it amounts to 74. We shall see in Subsection 3.3 that the maximum number of cycles that do not form a bundle in an RS-undecidable cactus is 5.

Therefore, in subsequent investigations on reflexive cacti two (already mentioned) conditions are imposed: (1) Cycles of cacti do not form a bundle and (2) cacti are RS-undecidable.

3.2. Starting investigation: bicyclic graphs with a bridge between cycles

As mentioned, the quest for reflexive cacti started with those with a bridge (i.e., a single edge) between two cycles. This case is tractable, yet still rather general. Let two cycles of arbitrary lengths be connected by a bridge whose vertices are c_1 and c_2. All maximal reflexive bicyclic graphs of that class have been found and described in [**25**].

Besides the 66 cases of maximal graphs in which at least one vertex of the cycles different from vertices c_1 and c_2 (Fig. 4(a)) is loaded (i.e. its degree is greater than 2), especially interesting part of the result is the case when both cycles are free (i.e. all vertices of cycles different from c_1 and c_2 are of degree 2).

Now, let c_1c_3 be an additional pendent edge (Fig. 4(a)). Let also S be a Smith tree and v any of its vertices dividing it into S_1 and S_2 (and both S_1 and S_2 keep a copy of v as in Fig. 3).

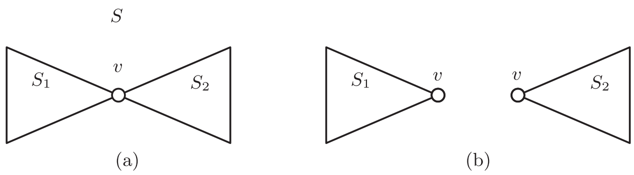

Figure 3.

If in an (RS-undecidable) bicyclic graph with a bridge between its cycles all vertices of the cycles except c_1 and c_2 are of degree 2, it is reflexive if and only if it is an induced subgraph of a graph formed by attaching S_1 and S_2 to c_2 and c_3 in all possible ways, as in Fig. 4(a), or of the graph of Fig. 4(b) for $\ell_1 = \ell_2 = 0$.

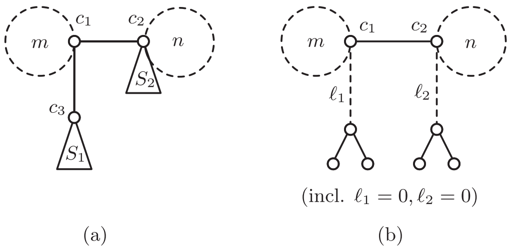

Figure 4.

Note that for this class of reflexive cacti, as well as for one class of tricyclic cacti, all minimal forbidden subgraphs are determined in [**13**].

3.3. Number of cycles

Since the previous case includes attaching a whole Smith tree to the vertex c_2, a simple generalization, when the Smith tree is replaced by a cycle, gives rise to the tricyclic maximal reflexive graph T_0 (Fig. 5), that is used in subsequent analysis.

Let us for a moment consider the general case of two bundles of cycles with a bridge that connects their common vertices. Let the first bundle have k cycles and the second one ℓ cycles. It is shown in [**24**] that for $\min(k, \ell) \geq 2$, it holds that $P(2) > 0$. Also, if $k = 1$ the graph T_0 shows that already by adding a single pendent edge at the vertex c_1, ℓ is at most 2, and if $\ell = 2$, adding any other pendent edge to the left cycle is not possible. If there are no pendent edges on the left cycle, the graph is RS-decidable. This shows that

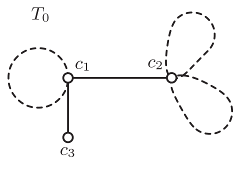

Figure 5.

the results of [**25**] cover all cases of RS-undecidable reflexive cacti with a bridge between its cycles. Therefore, the next step is the investigation of classes of cacti without a bridge between its cycles.

If we assume now that every cycle has at most two vertices belonging also to some other cycles, then the number of such vertices is at most 2. If there are 3 such vertices, then $\lambda_2 > 2$, by the RS-Theorem.

Now, if we consider a case where one cycle has at least three common vertices with other cycles, we say that this is the central cycle. The central cycle can at most be a quadrangle, since otherwise the graph has an RS-decidable subgraph.

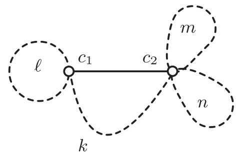

Figure 6.

In further quest for all reflexive cacti with 4 or more cycles it was natural to start from the graph in Fig. 6, which is a supergraph of T_0 for $k \geq 3$. Since $P(2) = 2\ell mn(k-3)$, the only possible values for k are 2 and 3.

Graph in Fig. 7(a) ($k = 3$) can be extended infinitely at the vertex c_4 preserving $P(2) = 0$ (extensions at c_3 are not possible because of T_0), but, at some point, no longer is $\lambda_2 = 2$, but $\lambda_3 = 2$ and $\lambda_2 > 2$. So, the task was to find this maximal extension that preserves $\lambda_2 = 2$. Similarly, starting from the graph in Fig. 2 and by adding a new vertex, we obtain the graph in Fig. 7(b). The equality $\lambda_2 = 2$ still holds, but further extensions are possible at d_1 and d_2.

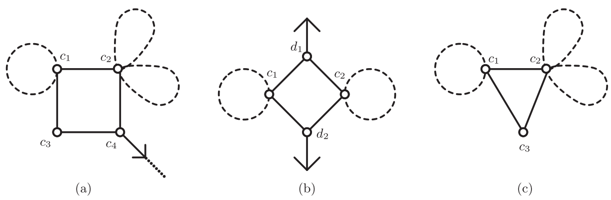

Figure 7.

Now, let us consider the maximum number of cycles. Evidently, T_0 allows a cycle at c_4, and we obtain a cactus with five cycles for which $\lambda_2 = 2$ holds. It is the graph denoted by Q_1 shown in Fig. 8(a). The only other possibility is to remove one of the outer cycles at c_2 and attach it to c_3, obtaining the graph Q_2 in Fig. 8(b) for which $\lambda_2 = 2$ holds.

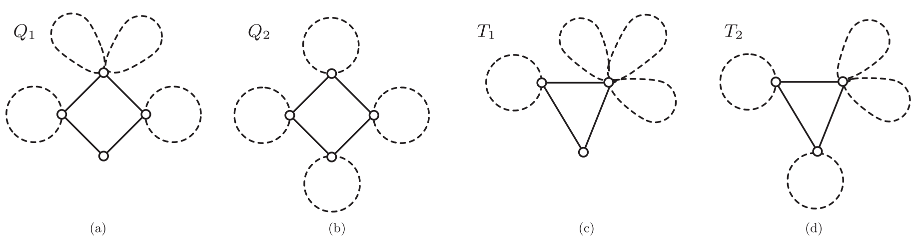

Figure 8.

If $k = 2$ (Fig. 7(c)), then $\lambda_2 < 2$. New cycles may be added only at c_2 and c_3 leading to graphs T_1 and T_2 shown in Fig. 8(c) and (d) for which $\lambda_2 = 2$ holds. All four resulting families of graphs are maximal reflexive cacti. Therefore, we conclude that a maximal RS-undecidable cactus whose cycles do not form a bundle has at most five cycles. The only such graphs with five cycles, which are all maximal, are the four families of graphs of Fig. 8 [**24**].

3.4. Reflexive cacti with four cycles

Once the maximum number of cycles, and the corresponding graphs, had been established, they were used as a starting point for further search for the maximal reflexive cacti with four cycles.

The next theorem, by Radosavljević, Mihailović, and Rašajski, shows how we can replace a cycle by a Smith tree in a maximal reflexive cactus of certain type.

Replacement Theorem. [**20**, **22**] *Let G be a maximal reflexive cactus which is the coalescence of a cycle C of an arbitrary length and a cactus K, with a common vertex v. Let $P_G(2) = 0$ and $P_K(2) < 0$ hold, and also let for any extension K_1 formed by attaching a pendent edge to K at any vertex $P_{K_1}(2) - 2P_{K_1-v}(2) > 0$ holds. If the free cycle C is replaced by an arbitrary Smith tree, attached to the vertex v in an arbitrary way, then the resulting graph is again a maximal reflexive cactus.*

Therefore, if the cycle at c_1 (or c_4) of Q_1 is replaced by a Smith tree, attached to c_1 (or c_4) at any vertex, all obtained graphs are maximal reflexive cacti (Fig. 9(a)). If one of the cycles at c_2 of the graph Q_1 is replaced by any of Smith trees, attached to c_2 at any vertex, all obtained graphs are maximal reflexive cacti (Fig. 9(b)). If any of the four non-central cycles of the graph Q_2 is replaced by any of Smith trees, attached to any vertex, all obtained graphs are maximal reflexive cacti (Fig. 9(c)). If we remove one of the four non-central cycles of Q_2, say the one attached to c_3, and attach parts of a Smith tree S_1 and S_2 to the vertices c_3 and c_2 (the same way as before) all obtained graphs are maximal reflexive cacti (Fig. 9(d)) [**21**, **24**].

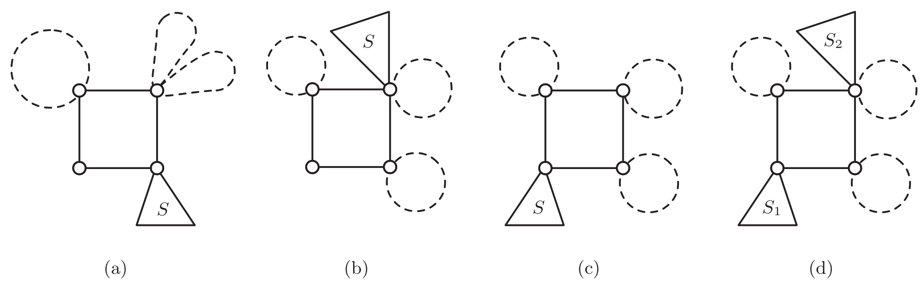

Figure 9.

Now, let us start from the graph T_1. If we remove one of the cycles at the vertex c_2 of the graph T_1, and attach parts of a Smith tree S_1 and S_2 to the vertices

c_3 and c_2 (the same way as before), all obtained graphs are maximal reflexive cacti, including cases when a whole Smith tree is attached to c_2 and c_3 (Fig. 10(a)).

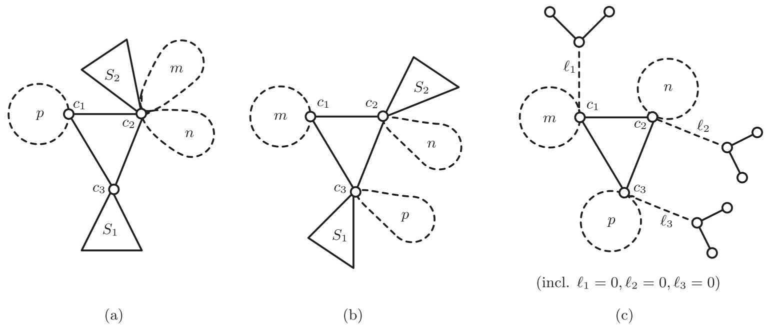

Figure 10.

Let G be a graph obtained by removing one of the cycles at the vertex c_2 of T_2 and attaching parts of a Smith tree S_1 and S_2 to the vertices c_3 and c_2 (the same way as before), including cases when a whole Smith tree is attached to c_3 or c_2 (Fig. 10(b)). Then, $\lambda_2(G) = 2$ and G is a maximal reflexive graph, with the exception of the case when, after removing c_1 from G, the remaining component with the bridge c_2c_3 is the graph of Fig. 4(b). In that case the corresponding maximal reflexive cactus is the one shown in Fig. 10(c) [**21**].

Now, if we remove one of the cycles at the vertex c_2 of the graph T_2, and attach some trees to all its c-vertices, such a graph is a maximal reflexive cactus if and only if it belongs to one of the ten families of graphs of Fig. 11 or the one of Fig. 10(c) [**21**].

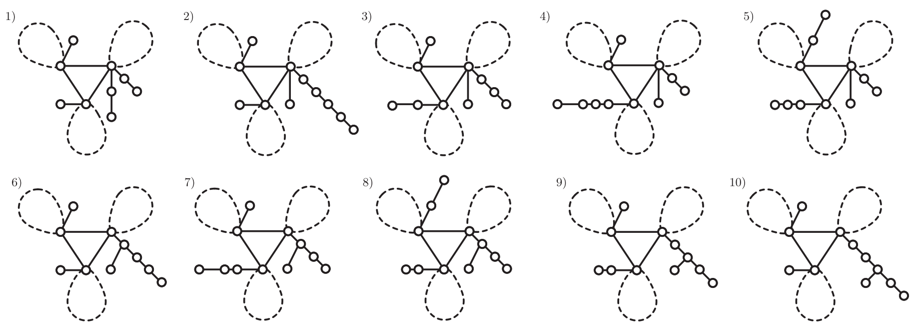

Figure 11.

Loading other vertices of non-central cycles also gives rise to some families of maximal reflexive cacti. These possibilities are discussed in [**23**] and completely solved by considering various particular cases.

An RS-undecidable cactus with four cycles whose cycles do not form a bundle and which, besides c-vertices, has at least one vertex of non-central cycles loaded, is reflexive if and only if it is an induced subgraph of some of the (families of) graphs

$H_1 - H_{48}$, $I_1 - I_9$, $J_1 - J_{11}$, $K_1 - K_{36}$, $L_1 - L_{12}$, $M_1 - M_{12}$ and $N_1 - N_{42}$ of [**23**, **24**].

3.5. Reflexive cacti with three cycles

Here we briefly cover the classes of tricyclic cacti. One of the cycles of the tricyclic cactus is the central cycle, and the other two are outer cycles, since these cycles do not form a bundle. If the two common vertices of cycles are not adjacent, then the central cycle has to be a quadrangle; otherwise it could not be reflexive, by the RS-Theorem. If the central cycle is at least a pentagon, then it is not possible to add more cycles in order for the graph to stay reflexive. But, if the central cycle is a triangle or a quadrangle, then it is possible to add new cycles, and therefore, we can easily see that these cycles could be replaced by Smith trees in order to get a tricyclic graph. There are four characteristic classes of tricyclic cacti and they are shown in Fig. 12.

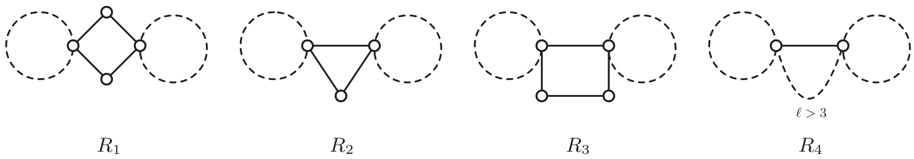

Figure 12.

The class R_1 was at first partially covered in [**15**], and then completely in [**13**]. Regarding the class R_2, each of the graphs of that class contains as a subgraph a bicyclic graph with a bridge between its cycles, and, because of that there are some limitation on which vertices could additionally be loaded in order for the graph to be reflexive [**25**]. Many of the graphs of class R_2 are described in [**13**]. Also, all the graphs of class R_3 are described in [**13**], and the same observation about the bicyclic graphs with a bridge holds in this case. In the case of R_4 for $\ell \geq 10$ no vertices could be additionally loaded, and all the graphs of class R_4 are described in [**26**].

3.6. Certain unicyclic reflexive graphs

The general problem of finding or describing all maximal reflexive unicyclic graphs seems intractable. It is clear from the RS-Theorem that the cycle in these graphs can be of an arbitrary length, also they can have an arbitrary number of vertices. Furthermore, they can have a vertex of arbitrary degree and, after its removal, the remaining graph can have an arbitrary number of components. This is why the search for maximal reflexive unicyclic graphs was directed towards some specific classes and it was natural to start from the classes of maximal reflexive cacti with two or more cycles, that had already been discovered. The main idea is to replace one or more cycles by a Smith tree. For example, by replacing the free cycle from a graph from the class of the bicyclic graphs with a bridge [**25**] by a Smith tree, we obtain a class of maximal reflexive unicyclic graphs. More on the role of Smith trees will be presented in the next section. Let us just make a note on the maximum number of loaded vertices in a maximal reflexive unicyclic graph. It is proved in [**20**] that the maximum number of loaded vertices of a maximal reflexive unicyclic graph is 8, and there are only 6 such graphs.

Regarding unicyclic reflexive graphs, it was shown in [**9**] that the length of the cycle of a unicyclic reflexive graph with seven loaded vertices is at most 10 and all such graphs with the length of the cycle 10, 9, and 8 were found.

4. ROLE OF SMITH GRAPHS

Here we give a brief review on the role of Smith graphs in investigations on reflexive cacti. These results were obtained simultaneously with those of the previous section. As we shall see, some of them improve the whole investigation by dividing the resulting graphs into specified subsets depending on certain operations based on Smith graphs.

4.1. Pouring of Smith trees

Besides cycles and whole Smith trees attached at some vertex of a graph, the first thing that may be noticed in observing the previous results is so-called pouring of Smith trees. This is the phenomenon that a Smith tree is divided at some vertex into two parts (Fig. 3(b)) and those parts are attached to two different vertices of a graph (as in Fig. 4(a)). As seen in the previous section, in that way a whole class of maximal bicyclic reflexive graphs had been described, with the exception of the graph in Fig. 4(b) for $\ell_1 = \ell_2 = 0$.

Additionally, these results extend to a class of maximal reflexive unicyclic graphs, when a cycle from the mentioned class of maximal bicyclic reflexive graphs with a bridge between its cycles was replaced by a Smith tree [**20**].

Pouring is also noticed in a class of maximal reflexive cacti with four cycles shown in Fig. 9(d).

It is important to say that even though graphs of Fig. 4(a) all have $\lambda_2 = 2$ and are maximal within the considered class of bicyclic graphs with a bridge, for their proper supergraphs illustrated in Fig. 9(d) $\lambda_2 = 2$ also holds and they are maximal within their class. But, it is worth saying that knowing the results on the maximum number of cycles in reflexive cacti, these graphs are maximal reflexive cacti not only within their class, but in the absolute sense.

Additionally, pouring of Smith trees is noticed in maximal reflexive cacti with four cycles given in Fig. 10(a) and 10(b), with exceptions when the Smith tree W_n is involved (Fig. 10(c)). We may add at this point an observation that W_n is often involved in exceptions.

4.2. Smith trees split into three parts

Let us now for a brief moment return to the other class of maximal reflexive cacti with four cycles, mentioned in the previous section and illustrated in Fig. 11. This class arises from the graph T_2 and in it we can easily see Smith trees

$S222$, $S313$ and $S215$ split into three parts and attached to the three vertices of the central triangle.

4.3. Pouring of pairs and triplets of Smith trees

Also, there is a phenomenon of pouring of pairs or triplets of Smith trees. As before, the path leading to these results involved removing some cycles and replacing them by Smith trees. To demonstrate that, we present here one class of maximal reflexive tricyclic cacti that arises from Q_1 (or Q_2). As proved in [**15**], the family of tricyclic graphs given in Fig. 11 represents a set of maximal reflexive cacti. Similarly, starting from graphs T_1 and T_2, we get maximal tricyclic reflexive cacti of Fig. 13(a). Within the class with this cyclic structure there are some exceptions that involve Smith tree W_n [**21**].

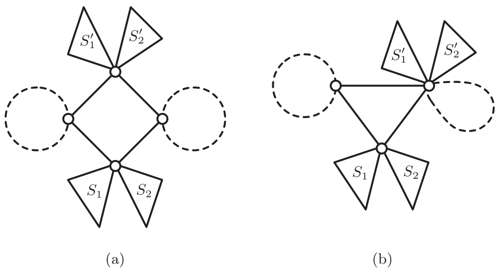

Figure 13.

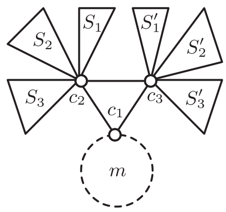

Figure 14.

Also, if we replace both outer cycles from the graph in Fig. 13(a) by a Smith tree, we get the corresponding unicyclic reflexive graphs, with the effect of pouring of pairs of Smith trees [**20**].

Let a bicyclic graph G consist of a cycle of an arbitrary length and a triangle, that have a common vertex c_1, and let triplets of Smith trees pour between the two remaining vertices of the triangle c_2 and c_3 (Fig. 14). Then G is a maximal reflexive cactus with some exceptions that, as before, often involve W_n [**22**].

Also, by replacing the free cycle from the graph in Fig. 14 by a Smith tree, we get the corresponding unicyclic reflexive graphs, with pouring of triplets of Smith trees [**20**].

4.4. σ – transformations

Now, let us focus again on maximal reflexive cacti with four cycles and demonstrate some other roles of Smith graphs in them. We examine extensions of some types of cactus without bridges by some modified Smith trees. The modifications of Smith trees are denoted by σ_i, $i = 1, 2, \ldots, 7$ [**27**], and the corresponding extensions are called σ_i–extensions. Here, σ_1 is a whole Smith tree, σ_2 is a Smith tree split into two parts, σ_3 is a Smith tree split into three parts, and we have just covered those cases. Next, σ_4 is a Smith tree with an added edge (Fig. 7(a)), σ_5 is a Smith tree with an added edge (Fig. 7(b)) and then split at one of the vertices of this new edge, σ_6 is a Smith tree with two vertices u and v identified (Fig. 7(c)), σ_7 is a Smith tree with two vertices u and v identified (Fig. 7(d)) and then split at that vertex into two parts. For the last four σ-transformations see Fig. 15.

Note that by adding an edge to a Smith tree or by identifying two vertices we obtain a cycle; thus corresponding cacti that we extend should be tricyclic in order to get cacti with four cycles.

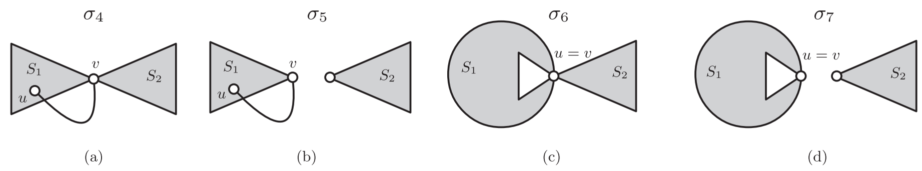

Figure 15.

Let us start from a tricyclic cactus G that consists of a triangle (with vertices d_1, d_2 and d_3) and two free cycles attached to d_1 and d_3 . Now we consider σ_4 and σ_5-extensions of the graph G. Attaching the graph σ_4 at its vertex v to one of the vertices d_1 or d_2 and the components of graph σ_5 to d_1 and d_2 of G we obtain graphs G_1, G_2, G_3 and G_4 in Fig. 16. They have four cycles. The fourth cycle stems from σ_4 or σ_5.

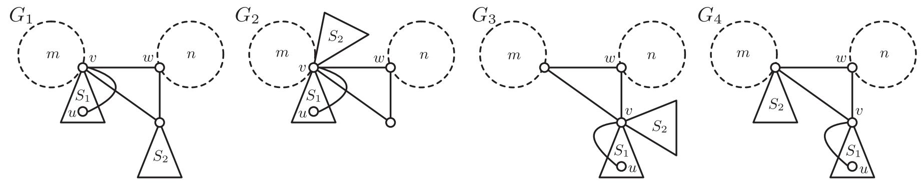

Figure 16.

σ_4 and σ_5-extensions of the tricyclic graph G lead to maximal reflexive cacti with four cycles. Maximal reflexive cacti with four cycles of type J, K, M, N from [**23**] that are at the same time graphs of type G_1, G_2, G_3 and G_4 are $J2$, $J4 - J10$; $K3$, $K7 - K8$, $K16 - K17$, $K20 - K21$, $K25 - K30$, $K33 - K34$; $M4$, $M7$, $M9 - M12$; $N7 - N8$, $N16 - N17$, $N20 - N21$, $N25 - N30$, $N33 - N34$. In graphs of type I and J from [**23**] the presence of Smith trees is apparent, but there are too few of them to make any kind of generalization.

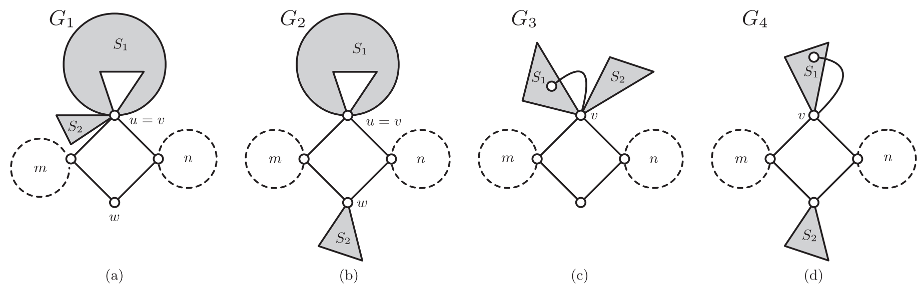

Figure 17.

Let now the starting graph be a tricyclic cactus G that consists of a quadrangle and two free cycles attached to its two opposite vertices. We consider σ_4, σ_5, σ_6, and σ_7-extensions of the tricyclic graph G to obtain maximal reflexive cacti with four cycles of type H from [**23**]. For the graph G we already have $\lambda_2 = 2$. Extending them we get maximal reflexive cacti within the given class and for many of them $\lambda_2 = \lambda_3 = 2$ holds. Mentioned σ-extensions produce graphs G_1, G_2, G_3 and G_4 in Fig. 17.

σ_4, σ_5, σ_6, and σ_7-extensions of the tricyclic graph G lead to maximal reflexive cacti with four cycles. Maximal reflexive cacti $H1$, $H5$, $H7$, $H9$, $H11$, $H30$, $H32$, $H33$, $H35$, $H36$ and $H38$ are graphs of type G_1, Fig. 17(a). Maximal reflexive cacti $H3$, $H10$, $H14$, $H16$, $H18$, $H23$, $H42$, $H43$, $H44$, $H45$ and $H48$ are graphs of type G_2, Fig. 17(b). Maximal reflexive cacti $H2$, $H4$, $H12$, $H19$, $H31$, $H35$, $H37$, $H39$ and $H41$ are graphs of type G_3, Fig. 17(c). Maximal reflexive cacti $H6$, $H8$, $H13$, $H24$, $H25$, $H26$, $H29$, $H46$ and $H47$ are graphs of type G_4, Fig. 17(d). Graphs $H15$, $H20$, $H21$, $H22$, $H34$ are supergraphs of some of the graphs of type (a), obtained by adding a pendent edge. Graphs $H17$, $H40$ and $H27$, $H28$ are subgraphs of the graphs of types (c) and (d), respectively, obtained by removing a pendent edge. (For corresponding graphs of type (c) and (d) $P_{G_i}(2) = 0$ holds, but $\lambda_2 > 2$ and $\lambda_3 = 2$.)

5. CONCLUSION

In the very beginnings, reflexive graphs were interesting for investigation only because of their connections with reflection groups. However, bound 2 for the second largest eigenvalue turned out to be very convenient for research. First, there exists a family of connected graphs whose index is 2 consisting of only six types of graphs (Smith graphs) and every other connected graph different from Smith graphs is comparable with them in the sense that it is either a proper subgraph of some Smith graph, or a proper supergraph of some of them. The existence of such family, for example, allows us to apply RS-Theorem very easily. Second, cycles are Smith graphs, so here we have a structural and a spectral property of graph apparently connected. For considering graphs with $\lambda_2 \leq r$ we see that $r = 2$ represents some kind of natural limit. For $r < 2$, if we consider a graph with a cut-vertex, we get that after removing a cut-vertex only one component can contain a cycle, while for $r > 2$ an investigation becomes extremely complex. Our case, where $r = 2$ is very demanding, but accessible. It is proved that the maximum number of cycles in RS-undecidable reflexive cacti is five and all such cacti with five and four cycles are completely described, by describing maximal graphs within those classes. The phenomenon of pouring of Smith trees, their pairs, triplets etc. is something that we met in almost every class of graphs we described. The idea of interpreting all results via Smith graphs arose and it led to investigating and describing transformations of Smith trees (σ–transformations) that happened in reflexive cacti with four cycles. Now it is clear that one can use this approach for reflexive cacti with less than 4 cycles.

APPENDIX

Here we survey determination of reflexive bipartite regular graphs (for short, RBR graphs) obtained by Koledin and Stanić [**10**].

Every regular bipartite graph (connected or not) of degree at most 2 is reflexive. These graphs are disjoint unions of complete graphs of order either 1 or 2, or disjoint unions of cycles of even orders. In addition, any disconnected RBR graph has degree at most 2 (since it is disconnected, its second largest eigenvalue is equal to its degree), so we can proceed to determine all connected RBR graphs. The set of all such graphs will be denoted by $\mathcal{R}$. Obviously, order n of any graph in $\mathcal{R}$ must be even. Moreover if the vertex degree of such a graph is at least $n-2$ (a complete graph or a cocktail-party graph), it is reflexive. Next, if we denote

$$\mathcal{R}^* = \left\{ G \in \mathcal{R},\ 3 \leq r \leq \frac{n}{2} \right\},$$

where r stands for vertex degree, then any graph in $\mathcal{R} \backslash \mathcal{R}^*$ is either a bipartite complement of a graph from $\mathcal{R}^*$ or it is necessarily reflexive. (A bipartite complement of a bipartite graph G is a bipartite graph with the same colour classes in which two vertices in different colour classes are adjacent precisely when they are non-adjacent in G; apart from the index and the least eigenvalue, the spectra of a bipartite graph and its bipartite complement coincide [**31**].) Therefore, it is sufficient to determine all graphs belonging to $\mathcal{R}^*$.

Using the structural considerations mostly based on the Interlacing Theorem and equitable partitions, Koledin and Stanić proved that the degree of graphs in $\mathcal{R}^*$ is at most 7, and that any of them has at most 30 vertices. The investigation is concluded by computer search, and the final result reads as follows:

(i) All regular bipartite graphs satisfying either $r \leq 2$ or $r \geq n-2$ are reflexive. If an RBR graph is disconnected then its degree is at most 2.

(ii) The set $\mathcal{R}^*$ consists of exactly 70 graphs. Precisely

- 20 graphs of degree 3,
- 30 graphs of degree 4,
- 6 graphs of degree 5,
- 9 graphs of degree 6,
- 5 graphs of degree 7.

(iii) Inspecting the obtained graphs, we conclude that 53 bipartite complements of graphs in $\mathcal{R}^*$ do not belong to $\mathcal{R}^*$, which means that the set $\mathcal{R}$ consists of exactly 123 graphs. In other words, there is an infinite family of RBR graphs described in (i), and additional 123 individual graphs.

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