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Some Notes on Minimal Self–Centered Graphs

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Abstract

A graph is called self–centered if all of its vertices have the same eccentricity. We prove some new properties of such graphs and obtain all minimal self–centered graphs of up to 10 vertices.

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1. Introduction

We restrict ourselves to undirected graphs without loops or multiple edges. Let G be an arbitrary graph and let $V(G)$ and $E(G)$ denote the sets of its vertices and edges, respectively. For vertices $v,\, w \in V(G)$, let $d(v,\, w)$ be the number of edges in a shortest path from v to w, called the distance between v and w. Let further $e(v) = max\{d(v,\, w) : w \in V(G)\}$ denote the eccentricity of the vertex v. The radius $r(G)$ and the diameter diam(G) are the minimum and maximum eccentricity, respectively. The set of all vertices $v \in V(G)$ with minimum eccentricity is known as the center of G. We say that the graph G is self–centered if all vertices have equal eccentricity. If self–centered graph G has diameter d, we simply say that G is d–self–centered. Some results on such graphs are established in [1], [2], [3], [4], [5], etc. Obviously, if graph is self–centered, every vertex belongs to its center and every vertex is the end of some diametral path. Also, a graph is 1–self–centered if and only if it is a complete graph.

2. On Self–Centered Graphs

The obtained family of minimal self–centered graphs (see the next section) enables us to formulate and prove some statements. In order to introduce some of them, we need to recall some notions (they are taken from [6], but usually used in other literature, as well). The union $G_1 \cup G_2$ of graphs $G_1(V_1,\, E_1)$ and $G_2(V_2,\, E_2)$ is the graph $G(V,\, E)$ where

$V = V_1 \cup V_2$ and $E = E_1 \cup E_2$. The complete product (or join) $G_1 \nabla G_2$ is obtained from $G_1 \cup G_2$ by joining every vertex of G_1 to every vertex of G_2 by an edge. The cartesian product $G_1 \times G_2$ is the graph with vertex set $V = V_1 \times V_2$ for which the adjacency of vertices is defined in the following way: two vertices (v_1, v_2), $(w_1, w_2) \in V$ are adjacent if and only if either $v_1 = w_1$ and $v_2w_2 \in E_2$ or $v_1w_1 \in E_1$ and $v_2 = w_2$. The line graph $L(G)$ of a given graph G is defined in the following way: there is a one–to–one correspondence between the set of vertices of $L(G)$ and the set of edges of G, and two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G are incident. We say that G is a minimal graph for some property if it loses this property when an arbitrary edge is removed.

First, we prove the following result.

Theorem 2.1. *Let G_1 and G_2 be graphs with sets of vertices and edges $V(G_1)$, $E(G_1)$ and $V(G_2)$, $E(G_2)$, respectively. Then*

(i) The complete product $G_1 \nabla G_2$ is a 2–self–centered graph if and only if $\Delta(G_1) < |V(G_1)| - 1$ and $\Delta(G_2) < |V(G_2)| - 1$.

(ii) If G_1 and G_2 are $m-$ and $n-$self–centered graphs respectively, then $G_1 \times G_2$ is $(m+n)-$self–centered graph. Reciprocally, if $G_1 \times G_2$ is self–centered graph then both graphs G_1 and G_2 are self–centered.

(iii) If G_1 is a minimal $d-$self–centered graph then every edge is contained in some diametral path.

Proof. (i) Let $G_1 \nabla G_2$ be 2–self–centered, then $e(v) = 2$ for every $v \in V(G_1 \nabla G_2)$, and therefore $\Delta(G_1) < |V(G_1)| - 1$ and $\Delta(G_2) < |V(G_2)| - 1$. Conversely if $\Delta(G_1) < |V(G_1)| - 1$ and $\Delta(G_2) < |V(G_2)| - 1$ then its complete product is clearly 2–self–centered.

(ii) This statement follows from the fact that the eccentricity of a vertex (v_1, v_2) is the sum of the eccentricities of v_1 and v_2.

(iii) If an edge e is not contained in any diametral path, then $G - e$ is d–self–centered, as well. But in this case G is not minimal self–centered. $\square$

Remark 2.1. *Statement (iii) of the previous theorem does not hold for an arbitrary self–centered graph. An example for this can be obtained by inserting any edge into graph G_7 depicted in Figure 2.*

Now we have the following theorem.

Theorem 2.2. *Let G and $L(G)$ be $d-$ and $f-$self–centered graphs, respectively. Then, $f \in \{d-1, d, d+1\}$.*

Proof. Let v_1v_2 and w_1w_2 be two arbitrary edges of G. We have $d(v_i, w_j) \le d$ for any choice of i, j. Therefore, the shortest path between these edges contains at most $d + 1$

vertices which implies $f \leq d+1$. Further, let $f < d-1$ and let the edges v_1v_2 and w_1w_2 correspond to vertices of $L(G)$ at distance f. Then, the eccentricity of any of vertices v_i, w_j $(i, j = 1, 2)$ is strictly less than d, which is a contradiction. □

If $G = K_2$ (1–self–centered) then $L(G) = K_1$ is 0–self–centered. On the other hand, examples of G and $L(G)$ which are both d–self–centered can be easily found. Hence we have the following interesting problem: *Determine the remaining (if any) d–self–centered graphs whose line graphs are $(d-1)$– or $(d{+}1)$–self–centered.*

Let $N(v)$ (resp. $N[v]$) denote the open (resp. closed) neighbourhood of an arbitrary vertex v. A self–centered graph does not contain cut–vertices or bridges (cut–edges). Also, the following theorem holds.

Theorem 2.3. *If G is d–self–centered, $(d \geq 2)$ then any maximal circuit in G consists of at least $2d$ vertices. Also, in the case $d = 2$, a maximal circuit has length 4 if and only if G is a complete bipartite graph with 2 vertices in one partition.*

Proof. The first part of the statement follows from the fact that any two vertices of a d–self–centered graph are contained in some common circuit (there are no cut–vertices).

If $G = K_{2,n-2}$, then any maximal circuit in G has length 4. On the other hand, let G be a 2–self–centered graph whose maximal circuits have 4 vertices. If $\Delta(G) = 2$, then $G = C_4 = K_{2,2}$. Now, suppose $\Delta \geq 3$ and v be a vertex such that $\deg(v) \geq 3$. Let $W = V(G)\backslash N[v]$. Since maximal circuit has length 4, $N(v)$ is independent. Further, the 2–self–centrality of G implies that every vertex $w \in W$ is adjacent to every vertex of $N(v)$. Finally, since the length of maximal circuit is 4, we get $|W| = 1$. Therefore, G is a complete bipartite graph with 2 vertices in one partition. □

In the following lemma we consider minimal self–centered graphs.

Lemma 2.1. *Let G be a 2–self–centered graph having $n \geq 5$ vertices and $2n-5$ edges. Then, no edge can be subdivided so that the resulting graph remains 2–self–centered.*

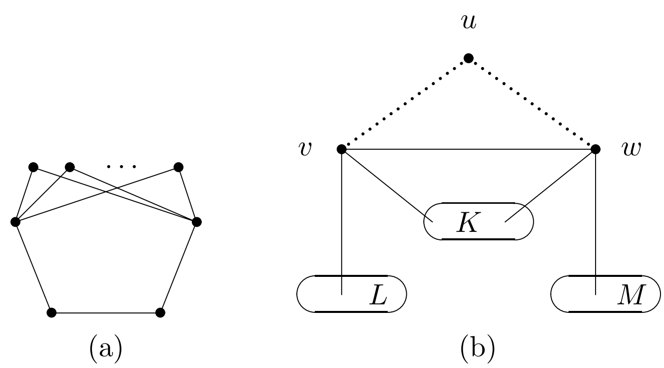

Figure 1

Proof First, a minimal 2–self–centered graph with $2n-5$ edges exists for every $n \geq 5$. Such a graph is not unique and it can have the form shown Figure 1 (a). Assume that an edge $vw \in E(G)$ can be subdivided by a new vertex u and that the obtained graph remains 2–self–centered. Since $e(u) = 2$, we have $N(v, w) = V(G)\backslash\{v, w\}$, where $N(v, w) = N(v) \cup N(w)$. Hence, $|N(v, w)| = n - 2$. Consider the following subsets of $V(G)$: $K = N(v) \cap N(w)$, $L = N(v)\backslash N[w]$ and $M = N(w)\backslash N[v]$ (compare Figure 1 (b)). Note that $|L|, |M| > 0$ (otherwise, we have $e(v) = 1$ or $e(w) = 1$). Let us count the number of edges of G. There are $2|K|$ edges which join every vertex from K to v and to w; $|L|$ (resp. $|M|$) edges which join every vertex from L (resp. M) to v (resp. w). Additionally, we need at least $|L| + |M| - 1$ edges, because of $d(l, m) \leq 2$, $l \in L$, $m \in M$. Finally, there is edge vw. Therefore,

$$\begin{aligned} 2n - 5 = |E(G)| &\geq 2|K| + 2(|L| + |M|) \\ &= 2(n - 2 - |L| - |M|) + 2(|L| + |M|) \\ &= 2n - 4. \end{aligned}$$

This contradiction proves the lemma. □

By using the previous lemma we obtain a short proof of one known result (see [2]):

Corollary 2.1. *If G is 2–self centered on $n \geq 5$ vertices then G has at least $2n-5$ edges.*

Proof. We have obtained the family of all minimal self–centered graphs of order up to 10 (Section 3) and we conclude that the statement holds for all these graphs. Now let $n > 10$ and we assume that the result holds for all graphs of order less than n. Let G be a minimal 2–self–centered graph on vertices with $|E(G)| \leq 2n - 6$. Then there is a vertex $u \in V(G)$, such that $\deg(u) = 2$ (otherwise, $|E(G)| \geq \frac{3n}{2} > 2n - 6$). Let v and w be the neighbours of u. If v and w are adjacent then the graph $G - u$ is 2–self–centered, as well and it has at most $2(n-1) - 6$ edges, which is a contradiction. On the other hand, if these vertices are not adjacent, then the graph $G' = (G - u) + vw$ is a 2–self–centered graph on $n - 1$ vertices with at most $2(n-1) - 5$ edges, which is a contradiction. □

3. Minimal Self–Centered Graphs

We obtain all minimal self–centered graphs up to order 10. A review of these graphs is given in the following table.

$n\backslash$radius	2	3	4	5
4	1	-	-	-
5	2	-	-	-
6	4	1	-	-
7	9	2	-	-
8	29	8	1	-
9	102	27	2	-
10	518	118	8	1

Table 1

This library helps us to formulate some of the statements in the previous section and it should be a useful tool for further research of this class of graphs. We use an algebraic way to obtain it. We first generate all connected graphs of order n. After that, we eliminate graphs which cannot be self–centered with specified radius (this includes the consideration of vertex degree, number of edges etc.). In order to single out the self–centered graphs from the reduced family, we consider the adjacency matrix A and its powers up to degree d (if we search for minimal d–self–centered graphs) and check the following conditions for these matrices: the $(i,\, j)$–entry, $i \neq j$, must be different from zero in at least one of these matrices and there is at least one such entry in every row (column) which equals zero in all matrices A, ..., A^{d-1}. In the graph whose adjacency matrix satisfies the previous conditions, we check its minimality by removing the edges. In this way we obtain all graphs in Table 1. Graph K_1 is 0–self–centered. Also, there are nine 1–self–centered graphs of order $n \leq 10$. If we add these values to the sum according to Table 1, we obtain exactly 843 graphs.

Here we present 2–self–centered graphs of order $n \leq 7$. The graphs are ordered lexicographically by their number of edges and by their spectra in non–decreasing order.

Additionally, all graphs from Table 1 are available on the following internet address http://www.matf.bg.ac.rs/∼zstanic/indexdiam.html. The obtained family enables us to better understand the structure of self–centered graphs and it will be a useful tool for further research.

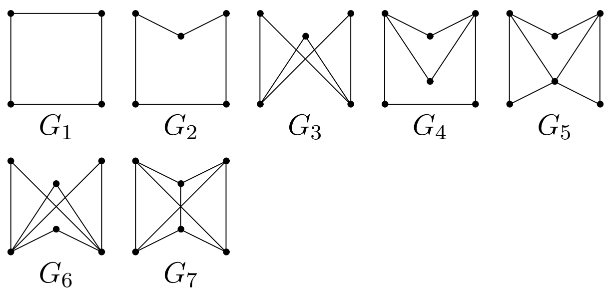

Figure 2

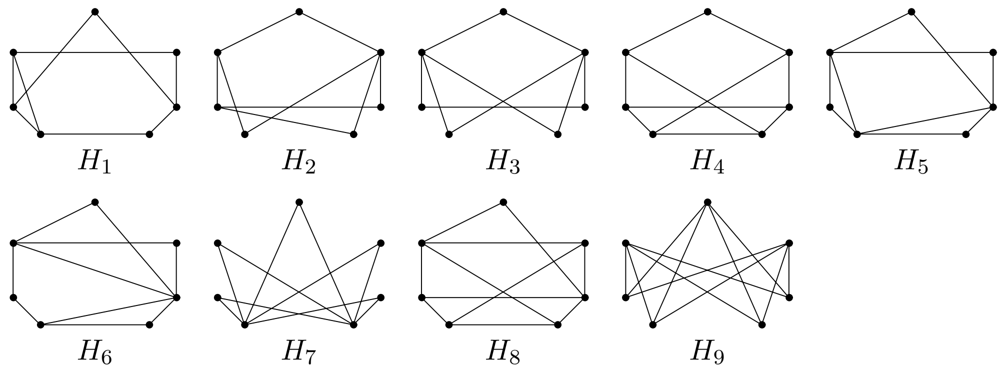

Figure 3

Acknowledgement

In order to generate the graphs of a specified order we use the Nauty library, so the author is thankful to its creator, professor Brendan McKay (see http://cs.anu.edu.au/∼bdm/nauty.html).

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