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On the spectrum of the net Laplacian matrix of a signed graph

by
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Abstract

Given a signed graph $\dot{G}$, let $A_{\dot{G}}$ and $D^{\pm}_{\dot{G}}$ be its standard adjacency matrix and the diagonal matrix of vertex net-degrees, respectively. The net Laplacian matrix of $\dot{G}$ is defined to be $N_{\dot{G}} = D^{\pm}_{\dot{G}} - A_{\dot{G}}$. In this paper we give some spectral properties of $N_{\dot{G}}$. We also point out some advantages and some disadvantages of using the net Laplacian matrix instead of the standard Laplacian matrix in study of signed graphs.

Key Words: (Net) Laplacian matrix, largest eigenvalue, join, graph product.
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1 Introduction

A *signed graph* $\dot{G}$ is a pair (G, σ), where $G = (V, E)$ is an ('unsigned') graph, called the *underlying graph*, and $\sigma\colon E \longrightarrow \{-1, +1\}$ is the *sign function*. The edge set of a signed graph is composed of subsets of positive and negative edges. Throughout the paper we interpret a graph as a signed graph with all the edges being positive. We denote the number of vertices of a signed graph by n.

The *degree* d_u of a vertex u of $\dot{G}$ is the number of its neighbours. The *positive degree* d^+_u is the number of positive neighbours of u (i.e., those adjacent to u by a positive edge). In the similar way, we define the *negative degree* d^-_u. The *net-degree* of u is defined to be $d^{\pm}_u = d^+_u - d^-_u$.

The adjacency matrix $A_{\dot{G}}$ of $\dot{G}$ is obtained from the standard adjacency matrix of its underlying graph by reversing the sign of all 1s that correspond to negative edges. The Laplacian matrix is defined to be $L_{\dot{G}} = D_{\dot{G}} - A_{\dot{G}}$, where $D_{\dot{G}}$ is the diagonal matrix of vertex degrees.

We define the *net Laplacian matrix* by $N_{\dot{G}} = D^{\pm}_{\dot{G}} - A_{\dot{G}}$, where $D^{\pm}_{\dot{G}}$ is the diagonal matrix of vertex net-degrees. (The name is suggested by Zaslavsky in private communication.) The adjacency matrix and the Laplacian matrix have received a great deal of attention in the theory of spectra of signed graphs. On the contrary, the net Laplacian appears very sporadically (under different names or without a name), but for example the authors of [2] pointed out its significance in study of controllability of undirected signed graphs. The purpose of this paper is to give more details on spectra of $N_{\dot{G}}$.

We denote the eigenvalues of $N_{\dot{G}} = (n_{i,j})$ by $\nu_1, \nu_2, \ldots, \nu_n$; of course, we include possible repetitions. We also assume that ν_1 is the largest eigenvalue and do not assume any ordering of the remaining ones. To ease language, we abbreviate the spectrum, the eigenvalues and the eigenvectors of $N_{\dot{G}}$ as the *spectrum*, the *eigenvalues* and the *eigenvectors* of $\dot{G}$.

Our results are a mixture of extensions of results concerning the Laplacian of graphs and the results describing the spectrum of the net Laplacian of specified signed graphs. In

Section 2 we give some additional terminology and notation. Spectral properties of $N_{\dot{G}}$ are considered in Section 3. In particular, we consider the behaviour of ν_1 under certain edge perturbations, derive some upper bounds for ν_1 and compute spectra of joins of signed graphs. In Section 4 we compute the net Laplacian, its eigenvalues and associated eigenvectors of some standard products of signed graphs. Finally, in Section 5 we give a detailed analysis of our results and emphasize some advantages of $N_{\dot{G}}$ with respect to $L_{\dot{G}}$, and vice versa.

2 Preparatory

We use $\mathbf{0}$ and $\mathbf{j}$ to denote the all-0 and the all-1 vector, respectively. The length may be given in the subscript. We say that a signed graph is *regular* if its underlying graph is regular. A signed graph is said to be *net-regular* if the net-degree is a constant on the vertex set.

The *negation* $-\dot{G}$ of a signed graph $\dot{G}$ is obtained by reversing the sign of every edge of $\dot{G}$. For signed graphs $\dot{G}_1$ and $\dot{G}_2$, the *positive join* (resp. *negative join*) $\dot{G}_1\nabla^+\dot{G}_2$ (resp. $\dot{G}_1\nabla^-\dot{G}_2$) is the signed graph obtained by adding all possible positive (resp. negative) edges between vertices of $\dot{G}_1$ and vertices of $\dot{G}_2$.

In the forthcoming Theorems 1 and 2, we deal with signed multigraphs, so those with multiple edges allowed. It is noteworthy to say that if $\dot{G}$ is such a multigraph, then the (u, v)-entry (for $u \neq v$) of $N_{\dot{G}}$ denotes the difference between the number of positive and the number of negative edges between u and v.

If $(x_1, x_2, \ldots, x_n)^\intercal$ is an eigenvector associated with an eigenvalue ν (of $N_{\dot{G}}$), then the *eigenvalue equation* related to ν at vertex u reads

$$(d_u^{\pm} - \nu)x_u = \sum_{uv \in E(\dot{G})} \sigma(uv)x_v. \tag{2.1}$$

Conversely, if (2.1) holds for some non-zero vector $\mathbf{x}$, real number ν and all the vertices of $\dot{G}$, then ν is an eigenvalue of $\dot{G}$ and $\mathbf{x}$ is an associated eigenvector.

To avoid possible confusion, we remark that a positive edge in $N_{\dot{G}}$ ($\dot{G}$ being a simple signed graph) is interpreted by -1, while a negative one is interpreted by 1. Accordingly, the diagonal entry is equal to the difference between the number of -1s and the number of 1s in the remainder of the corresponding row.

3 Spectral properties

We start with an extension of a result of Merris known as the *edge principle* [3].

Theorem 1. *Let ν be an eigenvalue associated with an eigenvector $\mathbf{x} = (x_1, x_2, \ldots, x_n)^\intercal$ of a signed multigraph $\dot{G}$. If, for some vertices u and v, we have $x_u = x_v$, then ν is an eigenvalue associated with the same eigenvector of a signed multigraph $\dot{G}'$ obtained by adding or removing (either a positive or a negative) edge between u and v.*

Proof. Using the eigenvalue equation for ν in $\dot{G}$, we get $(d_u^{\pm} - \nu)x_u = \sum_{ui \in E(\dot{G})} \sigma(ui)x_i$; so, the summation goes over all edges incident with u in a signed multigraph. By adding a positive (resp. negative) edge between u and v, we get $(d_u^{\pm}+1-\nu)x_u = x_v+\sum_{ui\in E(\dot{G})}\sigma(ui)x_i$ (resp. $(d_u^{\pm}-1-\nu)x_u = -x_v+\sum_{ui\in E(\dot{G})}\sigma(ui)x_i$), which is the eigenvalue equation for u in $\dot{G}'$. The eigenvalue equation for v in $\dot{G}'$ is obtained by interchanging the roles of u and v. Eigenvalue equations for the remaining vertices of $\dot{G}'$ are unchanged, so ν is the eigenvalue of $\dot{G}'$ and $\mathbf{x}$ is an associated eigenvector.

The case in which we remove an edge (if any) between u and v is proved in essentially the same way. □

We continue by considering the behaviour of the largest eigenvalue under certain edge perturbations.

Theorem 2. *Let $\dot{G}'$ be obtained from a signed multigraph $\dot{G}$ either by*

(i) reversing the sign of a negative edge,

(ii) adding a positive edge or

(iii) removing a negative edge,

then $\nu_1(\dot{G}) \leq \nu_1(\dot{G}')$, with equality if and only if for all the eigenvectors associated with $\nu_1(\dot{G})$ the coordinates which correspond to the endpoints of the edge in question are equal.

Proof. Denote the corresponding vertices by u and v and let $\mathbf{x} = (x_1, x_2, \dots, x_n)^{\intercal}$ be a unit eigenvector associated with $\nu_1(\dot{G})$. We have

$$\nu_1(\dot{G}) = \mathbf{x}^{\intercal} N_{\dot{G}}\mathbf{x} = \sum_{1\leq i,j\leq n} x_i n_{i,j} x_j,$$

which, for (i), can be written as

$$\nu_1(\dot{G}) = x_u^2 d_u^{\pm} + x_v^2 d_v^{\pm} + 2x_u x_v + \sum x_i n_{i,j} x_j,$$

where we simply extracted the three terms from the initial sum. Since

$$x_u^2 + x_v^2 - 2x_u x_v \geq 0, \tag{3.1}$$

we have

$$\begin{aligned}\nu_1(\dot{G}) &\leq x_u^2 d_u^{\pm} + x_v^2 d_v^{\pm} + 2x_u x_v + \sum x_i n_{i,j} x_j + 2(x_u^2 + x_v^2 - 2x_u x_v)\\ &= x_u^2(d_u^{\pm}+2) + x_v^2(d_v^{\pm}+2) - 2x_u x_v + \sum x_i n_{i,j} x_j\\ &= \sum_{1\leq i,j\leq n} x_i n'_{i,j} x_j \leq \max_{\mathbf{y}\in\mathbb{R}^n, ||\mathbf{y}||=1} \sum_{1\leq i,j\leq n} y_i n'_{i,j} y_j = \nu_1(\dot{G}'),\end{aligned}$$

where $(n'_{i,j})$ is the net Laplacian matrix of $\dot{G}'$ and, of course, the last equality follows by the Rayleigh principle.

The remaining two cases are proved in a very similar way; both proofs are based on the inequality (3.1), as well.

If $\nu_1(\dot{G}) = \nu_1(\dot{G}')$, then we have equality in (3.1), which gives $x_u = x_v$. Conversely, $x_u = x_v$ yields $\nu_1(\dot{G}) = \nu_1(\dot{G}')$ by Theorem 1. (Observe that case (i) of this theorem is also covered by the edge principle, as it can be realized in two steps: by removing a negative edge and adding a positive one to the same place). □

Here is a simple lemma.

Lemma 1. *For every signed graph $\dot{G}$:*

(i) $N_{\dot{G}} = -N_{-\dot{G}}$;

(ii) 0 *is an eigenvalue and* $\mathbf{j}$ *is an associated eigenvector;*

(iii) If $\dot{G}$ *is net-regular with net-degree* r*, then* $\nu_i(\dot{G}) = r - \lambda_i(\dot{G})$*, for* $1 \leq i \leq n$*, where* $\lambda_1, \lambda_2, \dots, \lambda_n$ *are the eigenvalues of the adjacency matrix* $A_{\dot{G}}$*;*

(iv) $\nu_1(\dot{G}) \leq \mu_1(\dot{G})$*, where* $\mu_1(\dot{G})$ *is the largest eigenvalue of the Laplacian matrix of* $\dot{G}$*;*

(v) $\nu_1(\dot{G}) \leq \nu_1(G)$*, where* G *is the underlying graph of* $\dot{G}$*;*

(vi) $\nu_1(\dot{G}) \leq 2\max_{1\leq i\leq n} d_i^+$*;*

(vii) If $\dot{G}$ *is decomposed into* k *edge-disjoint subgraphs* $\dot{G}_1, \dot{G}_2, \dots, \dot{G}_k$*, then we have* $\nu_1(\dot{G}) \leq \sum_{i=1}^{k} \nu_1(\dot{G}_i)$*.*

Proof. (i), (ii) and (iii) follow directly by definition of $N_{\dot{G}}$.

(iv): If $(x_1, x_2, \dots, x_n)^\intercal$ is a unit eigenvector associated with ν_1, then

$$\nu_1 = \sum_{1\leq i,j\leq n} x_i n_{i,j} x_j = \sum_{1\leq i,j\leq n} x_i l_{i,j} x_j - 2\sum_{i=1}^{n} x_i^2 d_i^-,$$

where $(l_{i,j})$ is the Laplacian matrix of $\dot{G}$. Thus, $\nu_1 = \sum_{1\leq i,j\leq n} x_i n_{i,j} x_j \leq \sum_{1\leq i,j\leq n} x_i l_{i,j} x_j$, and the result follows by the Rayleigh principle.

(v): This follows by Theorem 2(i), as G is obtained by reversing the sign of every negative edge.

(vi): By the Geršgorin circle theorem, we have $\nu_1(\dot{G}) \leq \max_{1\leq i\leq n}(d_i^{\pm} + d_i)$, which gives the result.

(vii): Since $N_{\dot{G}} = \sum_{i=1}^{k} N_{\dot{G}_i}$, we have

$$\begin{aligned}\nu_1(\dot{G}) &= \max_{\mathbf{x}\in\mathbb{R}^n,\,||\mathbf{x}||=1} \mathbf{x}^\intercal\Big(\sum_{i=1}^{k} N_{\dot{G}_i}\Big)\mathbf{x} = \max_{\mathbf{x}\in\mathbb{R}^n,\,||\mathbf{x}||=1} \sum_{i=1}^{k} \mathbf{x}^\intercal N_{\dot{G}_i}\mathbf{x}\\ &\leq \sum_{i=1}^{k} \max_{\mathbf{x_i}\in\mathbb{R}^n,\,||\mathbf{x_i}||=1} \mathbf{x_i}^\intercal N_{\dot{G}_i}\mathbf{x_i} = \sum_{i=1}^{k} \nu_1(\dot{G}_i),\end{aligned}$$

and we are done. □

By (i), all results concerning the largest eigenvalue of $\dot{G}$ can be reformulated for the least eigenvalue of $-\dot{G}$. Note that for (iii), r is an eigenvalue of $A_{\dot{G}}$ afforded by $\mathbf{j}$, which gives 0 in the spectrum of $N_{\dot{G}}$.

We proceed by another transfer from [3].

Theorem 3. *Let $\dot{G}_1$ and $\dot{G}_2$ be signed graphs with net Laplacian eigenvalues $\nu_1(\dot{G}_1), \nu_2(\dot{G}_1), \dots, \nu_{n_1}(\dot{G}_1) = 0$ and $\nu_1(\dot{G}_2), \nu_2(\dot{G}_2), \dots, \nu_{n_2}(\dot{G}_2) = 0$, respectively, and let $*$ be a fixed element of $\{+, -\}$. The eigenvalues of $\dot{G}_1 \nabla^* \dot{G}_2$ are $*(n_1 + n_2)$, $\nu_1(\dot{G}_1) * n_2$, $\nu_2(\dot{G}_1) * n_2, \dots, \nu_{n_1-1}(\dot{G}_1) * n_2$, $\nu_1(\dot{G}_2) * n_1$, $\nu_2(\dot{G}_2) * n_1, \dots, \nu_{n_2-1}(\dot{G}_2) * n_1$ and 0.*

If $\mathbf{x}$ is an eigenvector of $\dot{G}_1$ orthogonal to $\mathbf{j}$ and associated with an eigenvalue ν, then its extension defined to be zero on each vertex of $\dot{G}_2$ is an eigenvector of $\dot{G}_1 \nabla^* \dot{G}_2$ associated with $\nu * n_2$, and similarly for the eigenvectors of $\dot{G}_1 \nabla^* \dot{G}_2$ that arise from those of $\dot{G}_2$. The eigenvalue $*(n_1 + n_2)$ is associated with the eigenvector whose value is $-n_2$ on each vertex of $\dot{G}_1$ and n_1 on each vertex of $\dot{G}_2$.

Proof. Clearly, 0 is afforded by $\mathbf{j}$. For $\mathbf{y} = (\mathbf{x}^\intercal, \mathbf{0}^\intercal)^\intercal$, where $\mathbf{x}$ is an eigenvector described in the theorem, we have

$$N_{\dot{G}_1 \nabla^* \dot{G}_2} \mathbf{y} = \begin{pmatrix} (N_{\dot{G}_1} * n_2 I)\mathbf{x} \\ \mathbf{0} \end{pmatrix} = (\nu * n_2)\mathbf{y}.$$

Similarly, for $\mathbf{y} = (\underbrace{-n_2, -n_2, \dots, -n_2}_{n_1}, \underbrace{n_1, n_1, \dots, n_1}_{n_2})^\intercal$, we have

$$\begin{aligned} N_{\dot{G}_1 \nabla^* \dot{G}_2} \mathbf{y} &= \begin{pmatrix} (N_{\dot{G}_1} * n_2 I)(-n_2 \mathbf{j_{n_1}}) * n_1(-n_2 \mathbf{j_{n_1}}) \\ (N_{\dot{G}_2} * n_1 I)(n_1 \mathbf{j_{n_2}}) * n_2(n_1 \mathbf{j_{n_2}}) \end{pmatrix} \\ &= *\,(n_1 + n_2) \begin{pmatrix} -n_2 \mathbf{j_{n_1}} \\ n_1 \mathbf{j_{n_2}} \end{pmatrix} = *(n_1 + n_2)\mathbf{y}, \end{aligned}$$

since $N_{\dot{G}_i} \mathbf{j_{n_i}} = \mathbf{0}$, for $i \in \{1, 2\}$. □

Under the notation of Theorem 3, the multiplicity of 0 in the spectrum of $G_1 \nabla^+ G_2$ is equal to the multiplicity of $-n_2$ in the spectrum of G_1 plus the multiplicity of $-n_1$ in the spectrum of G_2 plus 1. The multiplicity of $n_1 + n_2$ is equal to the multiplicity of n_1 in the spectrum of G_1 plus the multiplicity of n_2 in the spectrum of G_2 plus 1. Analogously for $G_1 \nabla^- G_2$.

Here are more consequences.

Corollary 1. *For a signed graph $\dot{G}$ with n vertices, we have $\nu_1(\dot{G}) \le n$, with equality if and only if $\dot{G}$ is a positive join of two signed graphs.*

Proof. We exploit the well-known result of Kel'mans stating that $\nu_1(G) \le n$ (G being the underlying graph), with equality if and only if G is a (positive) join of two graphs, see [3, 4] or [5, p. 155]. The inequality follows by Lemma 1(v). If $\dot{G}$ is a positive join, then equality holds by Theorem 3. Conversely, if equality holds for some $\dot{G}$ which is not a positive join, then again by Lemma 1(v), we deduce that $\nu_1(G) \ge n$, which contradicts the mentioned result. □

Consequently, if $\dot{G}$ is a positive join, then $\nu_1(\dot{G}) = \nu_1(G)$. At the first glance, this equality might be surprising since G is obtained by a successive application of Theorem 2(i), but the explanation lies in Theorem 3 which gives an eigenvector associated with $\nu_1(\dot{G})$.

We proceed by the following lemma.

Lemma 2. *Let a signed graph $\dot{G}$ contain a set of vertices U, such that all of them are either (1) mutually non-adjacent, (2) adjacent by a positive edge or (3) adjacent by a negative edge, and let they share the same set of positive neighbours and the same set of negative neighbours outside U. Then, there exists an eigenvalue ν with multiplicity at least $|U|-1$, such that for (1) $\nu = d^{\pm}$, for (2) $\nu = d^{\pm}+1$ and for (iii) $\nu = d^{\pm}-1$, where $d^{\pm}$ denotes the common net-degree of vertices of U.*

Moreover, every eigenvector associated with some of the remaining eigenvalues is constant on U.

Proof. Clearly, we may assume that $|U| \geq 2$, since otherwise there is nothing to prove. Observe that a vector with exactly two non-zero coordinates – one equal to 1, the other equal to -1 – that correspond to any pair of vertices of U is an eigenvector associated with ν. Since the dimension of the corresponding eigenspace is $|U|-1$, the first part of the proof is completed.

Further, if $\mathbf{x}$ is associated with some of the remaining eigenvalues, then since $\mathbf{x}$ is orthogonal to all of the previously mentioned eigenvectors associated with ν, we get that it must be constant on U. □

Here is a particular case, where we can say more.

Corollary 2. *Under the notation of Lemma 2, if the vertices of U are of type (2) (resp. type (3)) and $\dot{G}$ is a positive (resp. negative) join of the signed graph induced by U and another signed graph, then $\nu = d^{\pm}+1$ (resp. $\nu = d^{\pm}-1$) and its multiplicity is at least $|U|$.*

Moreover, every eigenvector associated with some of the remaining non-zero eigenvalues is zero on U.

Proof. By Theorem 3, apart from the eigenvectors described in the proof of Lemma 2, the vector whose value is $-n-|U|$ on each vertex of U and $|U|$ on each vertex outside U is also associated with ν. In addition, this eigenvector is orthogonal to the mentioned ones, and the multiplicity of ν follows. Again, by Theorem 3, every non-zero eigenvalue is afforded by an eigenvector which is zero on U. □

4 Spectra of some standard products

In this section, let $\dot{G}_1$ be a signed graph with the vertex set $\{u_1, u_2, \ldots, u_{n_1}\}$, eigenvalues $\nu_1(\dot{G}_1), \nu_2(\dot{G}_1), \ldots, \nu_{n_1}(\dot{G}_1)$ and associated eigenvectors $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_{n_1}}$, and let $\dot{G}_2$ be a signed graph with the vertex set $\{v_1, v_2, \ldots, v_{n_2}\}$, eigenvalues $\nu_1(\dot{G}_2), \nu_2(\dot{G}_2), \ldots, \nu_{n_2}(\dot{G}_2)$ and associated eigenvectors $\mathbf{y_1}, \mathbf{y_2}, \ldots, \mathbf{y_{n_2}}$. We consider the Cartesian product $\dot{G}_1 \,\square\, \dot{G}_2$, the tensor product $\dot{G}_1 \times \dot{G}_2$ and the strong product $\dot{G}_1 \boxtimes \dot{G}_2$.

The set of vertices of any of them is identified as the Cartesian product of the sets of vertices of $\dot{G}_1$ and $\dot{G}_2$. In $\dot{G}_1 \,\square\, \dot{G}_2$, the vertices (u_i, v_j) and (u_k, v_l) are adjacent if and only if $u_i = u_k$ and $v_j \sim v_l$ or $u_i \sim u_k$ and $v_j = v_l$. The sign of an edge coincides with the sign of the edge it arises from. In $\dot{G}_1 \times \dot{G}_2$, the vertices (u_i, v_j) and (u_k, v_l) are adjacent if and only if $u_i \sim u_k$ and $v_j \sim v_l$. The sign of an edge is the product of signs of the corresponding edges of $\dot{G}_1$ and $\dot{G}_2$. In $\dot{G}_1 \boxtimes \dot{G}_2$, the vertices (u_i, v_j) and (u_k, v_l) are adjacent if and only if they are adjacent in any of the previous two products and the sign of an edge is determined as in the previous particular cases.

Lemma 3. *If $\dot{G}_1$ and $\dot{G}_2$ are above described signed graphs, then*

(i) $N_{\dot{G}_1 \Box \dot{G}_2} = N_{\dot{G}_1} \otimes I_{n_2} + I_{n_1} \otimes N_{\dot{G}_2}$,

(ii) $N_{\dot{G}_1 \times \dot{G}_2} = D^{\pm}_{\dot{G}_1} \otimes N_{\dot{G}_2} + N_{\dot{G}_1} \otimes D^{\pm}_{\dot{G}_2} - N_{\dot{G}_1} \otimes N_{\dot{G}_2}$,

(iii) $N_{\dot{G}_1 \boxtimes \dot{G}_2} = N_{\dot{G}_1 \Box \dot{G}_2} + N_{\dot{G}_1 \times \dot{G}_2}$,

where $\otimes$ denotes the standard Kronecker product and $D^{\pm}_{\dot{G}_i}$ is the diagonal matrix of vertex net-degrees of $\dot{G}_i$, for $i \in \{1, 2\}$.

Proof. (i): The result follows since $N_{\dot{G}_1 \Box \dot{G}_2}$ can be considered as the $n_1 \times n_1$ block matrix with blocks of size $n_2 \times n_2$, such that the ith diagonal block is $d^{\pm}_{u_i} N_{\dot{G}_2}$ and, for $1 \leq i \neq j \leq n_1$ the (i, j)-block is the product of the (i, j)-entry of $N_{\dot{G}_1}$ and I_{n_2}.

(ii): The left-hand side of the identity under consideration is equal to $D^{\pm}_{\dot{G}_1} \otimes D^{\pm}_{\dot{G}_2} - A_{\dot{G}_1} \otimes A_{\dot{G}_2}$, which follows by considering the same blocking as before. On the other hand, by computing the right-hand side, we get $D^{\pm}_{\dot{G}_1} \otimes N_{\dot{G}_2} + N_{\dot{G}_1} \otimes D^{\pm}_{\dot{G}_2} - N_{\dot{G}_1} \otimes N_{\dot{G}_2} = D^{\pm}_{\dot{G}_1} \otimes \big(D^{\pm}_{\dot{G}_2} - A_{G_2}\big) + \big(D^{\pm}_{\dot{G}_1} - A_{G_1}\big) \otimes D^{\pm}_{\dot{G}_2} - \big(D^{\pm}_{\dot{G}_1} - A_{\dot{G}_1}\big) \otimes \big(D^{\pm}_{\dot{G}_2} - A_{\dot{G}_2}\big) = D^{\pm}_{\dot{G}_1} \otimes D^{\pm}_{\dot{G}_2} - A_{\dot{G}_1} \otimes A_{\dot{G}_2}$, and we are done.

(iii): This follows by definition of the strong product. ∎

We compute the eigenvalues.

Theorem 4. *If $\dot{G}_1$ and $\dot{G}_2$ are above described signed graphs, then*

(i) the eigenvalues of $N_{\dot{G}_1 \Box \dot{G}_2}$ are $\nu_i(\dot{G}_1) + \nu_j(\dot{G}_2)$, $1 \leq i \leq n_1$, $1 \leq j \leq n_2$;

(ii) if $\dot{G}_1$ and $\dot{G}_2$ are net-regular with net-degrees r and s, respectively, then the eigenvalues of $N_{\dot{G}_1 \times \dot{G}_2}$ are $r\nu_j(\dot{G}_2) + s\nu_i(\dot{G}_1) - \nu_i(\dot{G}_1)\nu_j(\dot{G}_2)$, $1 \leq i \leq n_1$, $1 \leq j \leq n_2$;

(ii) if $\dot{G}_1$ and $\dot{G}_2$ are net-regular with net-degrees r and s, respectively, then the eigenvalues of $N_{\dot{G}_1 \times \dot{G}_2}$ are $(r+1)\nu_j(\dot{G}_2) + (s+1)\nu_i(\dot{G}_1) - \nu_i(\dot{G}_1)\nu_j(\dot{G}_2)$, $1 \leq i \leq n_1$, $1 \leq j \leq n_2$.

In all cases, the corresponding eigenvectors are $\mathbf{x_i} \otimes \mathbf{y_j}$, $1 \leq i \leq n_1$, $1 \leq j \leq n_2$.

Proof. The result follows since the eigenvalues of $A \otimes B$ are all the possible products of an eigenvalue of A and an eigenvalue of B, and the corresponding eigenvectors are the Kronecker products of the eigenvectors of A and B. For (ii) and (iii) we also use the net-regularity argument. ∎

5 Comments

By giving a detailed analysis of the foregoing results, we emphasize some advantages of using the net Laplacian instead of the standard Laplacian in study of signed graphs. They are mostly based on fact that some results concerning the Laplacian of ('unsigned') graphs can be transferred to the net Laplacian, but cannot be transferred to the Laplacian of signed graphs. Needless to add, if some result holds for the net Laplacian, then it also holds in particular case of the Laplacian of graphs.

+ Theorem 2 is an extension of a well-known result for the Laplacian of graphs (expressed in part (ii)). It does not hold for the Laplacian of signed graphs in its parts (i) and (iii).

+ The statement (i) of Lemma 1 does not hold for $L_{\dot{G}}$, (ii) holds if and only if $\dot{G}$ is balanced (see [1] for details), (iii) holds if and only if $\dot{G}$ is regular, so for a narrowed class, (v) and (vi) do not hold for $\mu_1(\dot{G})$ (which follows easily) and (vii) holds for $L_{\dot{G}}$, as well.

+ Theorem 3 is an extension of a result for the Laplacian of graphs [3]. It does not hold for the Laplacian of signed graphs, and the same applies for Corollary 1. Counterexamples can be constructed by hand. It is worth mentioning that there is no similar result for the adjacency matrix of (signed) graphs, either. This gives a credit to the Laplacian of graphs and the net Laplacian (of signed graphs) in study of spectra of joins.

+ Lemma 2 can be formulated in a similar form for the Laplacian of signed graphs, but the subsequent Corollary 2 cannot since it relies on Theorem 3.

+ Lemma 3 holds for the Laplacian of signed graphs, as well. We did not find this result in literature, but in fact it is an easy exercise. The same holds for (i) of Theorem 4, but (ii) and (iii) hold if and only if $\dot{G}_1$ and $\dot{G}_2$ are regular.

Of course, there are some disadvantages. We point out just the three general ones.

- Contrary to $L_{\dot{G}}$, the matrix $N_{\dot{G}}$, in general, is not positive semidefinite, so it may have positive and negative eigenvalues. Positive definiteness of the former matrix is frequently exploited in literature.

- If R is the vertex-edge incidence matrix of G, then $L_G = RR^\intercal$. The 'signed' counterpart to R_G is the vertex-edge incidence matrix formed on the basis of so-called vertex-edge bi-orientation η, and it is usually denoted by B_η. In this case we have $L_{\dot{G}} = B_\eta B_\eta^\intercal$. The details can be found in [6]. There is a number of results and concepts that rely on the identity $L_G = RR^\intercal$, and many of them can be transferred to $L_{\dot{G}}$ with B_η in the role of R. Unfortunately, according to our knowledge, there is no something similar for $N_{\dot{G}}$.

- If U is a subset of the vertex set of $\dot{G}$, then by $\dot{G}^U$ we denote the signed graph obtained by reversing the sign of every edge with exactly one end in U. We say that $\dot{G}^U$ is *switching equivalent* to $\dot{G}$. Switching equivalence is one of the fundamental concepts in the theory of signed graphs. It is easy to conclude that switching equivalent signed graphs share the same spectrum of the adjacency matrix and that of the Laplacian matrix. Unfortunately, this fails to hold in the case of the spectrum of the net Laplacian matrix.

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