ON GRAPHS WITH UNICYCLIC STAR COMPLEMENT FOR 1 AS THE SECOND LARGEST EIGENVALUE *

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The star complement technique is a spectral tool recently developed for constructing some bigger graphs from their smaller parts, called star complements. Here we first identify among unicyclic graphs those graphs which can be star complements for 1 as the second largest eigenvalue. Using the graphs obtained, we next search for their maximal extensions, either by theoretical means, or by computer aided search.

1. Introduction

We will consider only simple graphs, that is finite, undirected graphs without loops or multiple edges. If G is such a graph with vertex set $V_G = \{1, 2, \dots, n\}$, the *adjacency matrix* of G is $n \times n$ matrix $A_G = (a_{ij})$, where $a_{ij} = 1$ if there is an edge between the vertices i and j, and 0 otherwise. The *eigenvalues* of G, denoted by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, are just the eigenvalues of A_G. Note, the eigenvalues of G are real and do not depend on vertex labelling. Additionally, for connected graphs $\lambda_1 > \lambda_2$ holds. The

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characteristic polynomial of G is the characteristic polynomial of its adjacency matrix, so $P_G(\lambda) = \det(\lambda I - A_G)$. For more details on graph spectra, see [4].

If μ is an eigenvalue of G of multiplicity k, then a *star set* for μ in G is a set X of k vertices taken from G such that μ is not an eigenvalue of $G - X$. The graph $H = G - X$ is then called a *star complement* for μ in G (or a *μ-basic subgraph* of G in [8]). (Star sets and star complements exist for any eigenvalue and any graph; they need not be unique.) The H-neighborhoods of vertices in X can be shown to be non-empty and distinct, provided that $\mu \notin \{-1, 0\}$ (see [6], Chapter 7). If $t = |V_H|$, then $|X| \leq \binom{t}{2}$ (see [1]) and this bound is best possible.

It can be proved that if Y is a proper subset of X then $X - Y$ is a star set for μ in $G - Y$, and therefore H is a star complement for μ in $G - Y$. If G has star complement H for μ, and G is not a proper induced subgraph of some other graph with star complement H for μ, then G is a *maximal graph* with star complement H for μ, or it is an *H-maximal graph* for μ. By the above remarks, there are only finitely many such maximal graphs, provided $\mu \notin \{-1, 0\}$. In general, there will be only several maximal graphs, possibly of different orders, but sometimes there is a unique maximal graph (if so, this graph is characterized by its star complement for μ).

We now mention some results from the literature in order to make the paper more self–contained (they are taken from [5, 6, 7]).

The following result is known as the Reconstruction Theorem (see, for example, [6], Theorems 7.4.1 and 7.4.4).

Theorem 1.1 *Let G be a graph with adjacency matrix*

$$\begin{pmatrix} A_X & B^T \\ B & C \end{pmatrix},$$

where A_X is the adjacency matrix of the subgraph induced by the vertex set X. Then X is a star set for μ if and only if μ is not an eigenvalue of C and $\mu I - A_X = B^T(\mu I - C)^{-1}B$.

From the above, we see that if μ, C and B are fixed then A_X is uniquely determined. In other words, given the eigenvalue μ, a star complement H for μ, and the H–neighborhoods of the vertices in the star set X, the graph G is uniquely determined. In the light of these facts, we may next ask to what extent G is determined only by H and μ. Having in mind the observation above, it is sufficient to consider graphs G which are H-maximal for μ.

Following [2], we will now fix some further notation and terminology. Given a graph H, a subset U of $V(H)$ and a vertex u not in $V(H)$, denote by $H(U)$ the graph obtained from H by joining u to all vertices of U. We will say that u (U, $H(U)$) is a *good vertex* (resp. *good set*, *good extension*) for μ and H, if μ is an eigenvalue of $H(U)$ but is not an eigenvalue of H. By Theorem 1.1, a vertex u and a subset U are good if and only if $\mathbf{b}_u^T(\mu I - C)^{-1}\mathbf{b}_u = \mu$, where $\mathbf{b}_u$ is the characteristic vector of U (with respect to $V(H)$) and C is the adjacency matrix of H. Assume now that U_1 and U_2 are not necessarily good sets corresponding to vertices u_1 and u_2, respectively. Let $H(U_1, U_2; 0)$ and $H(U_1, U_2; 1)$ be the graphs obtained by adding to H both vertices, u_1 and u_2, so that they are non-adjacent in the former graph, while adjacent in the latter graph. We say that u_1 and u_2 are *good partners* and that U_1 and U_2 are *compatible sets* if μ is an eigenvalue of multiplicity two either in $H(U_1, U_2; 0)$ or in $H(U_1, U_2; 1)$. (Note, if $\mu \notin \{-1, 0\}$, any good set is non-empty, any two of them if corresponding to compatible sets are distinct; see [6], cf. Proposition 7.6.2.) By Theorem 1.1, two vertices u_1 and u_2 are good partners (or two sets U_1 and U_2 are compatible) if and only if $\mathbf{b}_{u_1}^T(\mu I - C)^{-1}\mathbf{b}_{u_2} \in \{-1, 0\}$, where $\mathbf{b}_{u_1}$ and $\mathbf{b}_{u_2}$ are defined as above. In addition, it follows (again by Theorem 1.1) that any vertex set X in which all vertices are good, both individually and in pairs, gives rise to a *good extension*, say G, in which X can be viewed as a star set for μ, while H as the corresponding star complement.

The above considerations shows us how we can introduce a technique, also called a *star complement technique*, for finding (or constructing) graphs with certain spectral properties. In this context the graphs we are interested in should have some prescribed eigenvalue usually of a very large multiplicity. If G is a graph in which μ is an eigenvalue of multiplicity $k > 1$, then G is a good (k-vertex) extension of some of its star complements, say H (in particular, G is H-maximal for μ). The *star complement* technique consists of the following: In order to find H-maximal graphs for μ ($\neq -1, 0$), we form an *extendability graph* whose vertices are good vertices for μ and H, and add an edge between two good vertices whenever they are good partners. Now it is easy to see that the search for maximal extensions is reduced to the search for maximal cliques in the extendability graph (see, for example, [5, 7]). Of course, among H-maximal graphs some of them can be mutually isomorphic.

Connected graphs with $\lambda_2 \leq 1$ were not too much studied in the literature.

Some known results are related to bipartite graphs and (generalized) line graphs (see [9] for details). The star complement technique in this context is, for the first time, used in [11]. In that paper it was taken that star complements are trees or complete graphs. Here we focus our attention on unicyclic graphs (i.e. connected graph having the same order and size) in the role of star complements for $\mu = 1$ as the second largest eigenvalue.

2. Main results

In what follows let H (in the role of star complement for $\mu = 1$ as the second largest eigenvalue) be a unicyclic graph. If not told otherwise, C stands for the unique cycle of H. Note also that $\lambda_2(H) < 1$ - by the Interlacing Theorem (see [4], p. 19). In addition, we also have that $\lambda_2(G) = 1$, where G is an arbitrary good extension of H for $\mu = 1$.

Lemma 2.1 *Under the above assumptions on H, the length of C is at most five.*

Proof. For a cycle C_n with $n \geq 6$ we have $\lambda_2(C_n) \geq 1$. Therefore, such a cycle cannot be a star complement for $\lambda_2 = 1$. In addition, by the Interlacing Theorem, any unicyclic graph containing cycle C_n with $n \geq 6$, cannot be a star complement for $\lambda_2 = 1$, and the proof follows.

Lemma 2.2 *Under the above assumptions on H, any vertex of H is at distance at most one from C.*

Proof. Observe first that P_n with $n \geq 5$ (by the same arguments as above) cannot be a star complement for $\lambda_2 = 1$. Therefore, as already noticed in [11], Lemma 3.1, the diameter of H is at most three. If some vertex of H is at distance two (or more) from C then the diameter is greater than three unless H consists of a triangle having a path of length two attached at some vertex of the triangle (and possibly some hanging edges attached at the same vertex). But in the latter situation 1 is the second largest eigenvalue of the resulting graph(s), and the proof follows.

The following theorem gives a characterization of all unicyclic graphs which can be star complements for $\lambda_2 = 1$.

Theorem 2.1 *A unicyclic graph H is a star complement for $\lambda_2 = 1$ if and only if it is one of the graphs depicted in Fig. 1.*

Sketch proof. We first determine all unicyclic graphs with $\lambda_2 < 1$. By Lemmas 2.1 and 2.2, we have to consider only those graphs containing a cycle C (of length ≤ 5) and possibly some hanging edges attached at the vertices of C. By direct calculations, we find that only the graphs H_1, H_3, H_4, H_8 and H_9 are maximal under the above constraints. In addition, H_{10} is not maximal for a fixed $n \geq 4$, but satisfies all imposed constraints. This can be verified for any n by making use of the Interlacing Theorem after removing a vertex labeled by 1; this gives that $\lambda_2(H_{10}) \leq 1$. On the other hand $\lambda_2(H_{10}) \neq 1$ since $P_{H_{10}}(1) = -4$ for any $n \geq 1$, as can be easily shown by making use of the Heilbronner formula (see [4] p. 59). We next have that, except for the graphs of Fig. 1 and two additional graphs, one being C_4 (the subgraph of H_2) and the other C_3 (the subgraph of H_5), there are no more graphs of interest. In what remains we have to see which of these graphs afford good extensions for $\mu = 1$. Firstly, for every graph of Fig. 1 except H_{10} we can find (by direct computations) at least one good set; this can be done by calculations for H_{10} (in addition, all good sets are given in the next theorem). Secondly, it is easy to check (say by a brute force computation) that the remaining two graphs do not contain any good set. This completes the proof.

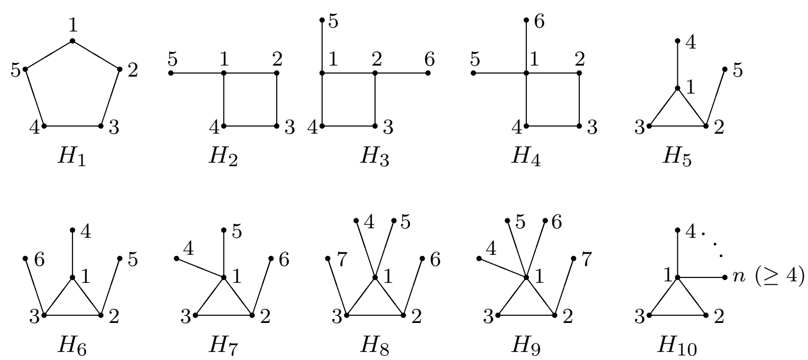

Figure 1. Unicyclic star complements for $\mu = 1$

We now proceed to identify all good sets U for $\mu = 1$ in all graphs $H_1 - H_{10}$ of Fig. 1. For this aim we will use either a brute force computations, or calculations based on the Schwenk formula (see [4] p. 78) which can be stated as follows: For a given (simple) graph G, let $\mathcal{C}(v)$ denote the set of all cycles containing a vertex v of G. Then

$$P_G(\lambda) = \lambda P_{G-v}(\lambda) - \sum_{w \sim v} P_{G-v-w}(\lambda) - 2 \sum_{C \in \mathcal{C}(v)} P_{G-V(C)}(\lambda), \qquad (1)$$

where $w \sim v$ denotes that w is a vertex adjacent to v, while $G - V(C)$ is the graph obtained from G by removing all vertices belonging to the cycle C (note also that $P_H(\lambda) = 1$ if H is an empty graph).

Theorem 2.2 *For any graph H_i $(i = 1, 2, \ldots, 10)$ of Fig. 1 its good sets are as given in the following list:*

$H_1 : U_1 = \{1\},\ U_2 = \{2\},\ U_3 = \{3\},\ U_4 = \{4\},\ U_5 = \{5\}, U_6 = \{1, 2, 3\},$
$U_7 = \{1, 2, 5\}, U_8 = \{1, 4, 5\},\ U_9 = \{2, 3, 4\},\ U_{10} = \{3, 4, 5\};$

$H_2 : U_1 = \{3\},\ U_2 = \{2, 5\},\ U_3 = \{4, 5\},\ U_4 = \{2, 3, 4\};$

$H_3 : U_1 = \{1\},\ U_2 = \{2\},\ U_3 = \{3\},\ U_4 = \{4\},\ U_5 = \{1, 6\},\ U_6 = \{2, 5\},$
$U_7 = \{3, 5\}, U_8 = \{4, 6\},\ U_9 = \{1, 3, 4\},\ U_{10} = \{2, 3, 4\},$
$U_{11} = \{3, 5, 6\},\ U_{12} = \{4, 5, 6\}, U_{13} = \{1, 2, 3, 6\},\ U_{14} = \{1, 2, 4, 5\};$

$H_4 : U_1 = \{1\},\ U_2 = \{2\},\ U_3 = \{4\},\ U_4 = \{2, 5\},\ U_5 = \{2, 6\},$
$U_6 = \{3, 5\},\ U_7 = \{3, 6\}, U_8 = \{4, 5\},\ U_9 = \{4, 6\},\ U_{10} = \{1, 2, 3\},$
$U_{11} = \{1, 3, 4\},\ U_{12} = \{3, 5, 6\}, U_{13} = \{1, 2, 4, 5, 6\};$

$H_5 : U_1 = \{3, 4\},\ U_2 = \{3, 5\},\ U_3 = \{1, 4, 5\},\ U_4 = \{2, 4, 5\};$

$H_6 : U_1 = \{1, 5\},\ U_2 = \{1, 6\},\ U_3 = \{2, 4\},\ U_4 = \{2, 6\},\ U_5 = \{3, 4\},$
$U_6 = \{3, 5\}, U_7 = \{1, 4, 5, 6\},\ U_8 = \{2, 4, 5, 6\},\ U_9 = \{3, 4, 5, 6\};$

$H_7 : U_1 = \{2\},\ U_2 = \{3, 4\},\ U_3 = \{3, 5\},\ U_4 = \{1, 4, 6\},\ U_5 = \{1, 5, 6\},$
$U_6 = \{2, 4, 5\}, U_7 = \{3, 4, 6\},\ U_8 = \{3, 5, 6\},\ U_9 = \{1, 2, 3, 6\},$
$U_{10} = \{2, 4, 5, 6\};$

$H_8 : U_1 = \{1\},\ U_2 = \{2\},\ U_3 = \{3\},\ U_4 = \{1, 6\},\ U_5 = \{1, 7\},$
$U_6 = \{2, 4\},\ U_7 = \{2, 5\}, U_8 = \{3, 4\},\ U_9 = \{3, 5\},\ U_{10} = \{1, 6, 7\},$
$U_{11} = \{2, 4, 7\},\ U_{12} = \{2, 5, 7\},\ U_{13} = \{3, 4, 6\},\ U_{14} = \{3, 5, 6\},$
$U_{15} = \{1, 2, 3, 6\},\ U_{16} = \{1, 2, 3, 7\},\ U_{17} = \{1, 4, 6, 7\},$
$U_{18} = \{1, 5, 6, 7\},\ U_{19} = \{2, 4, 5, 7\},\ U_{20} = \{3, 4, 5, 6\},$
$U_{21} = \{1, 2, 3, 4, 5\},\ U_{22} = \{2, 4, 5, 6, 7\},\ U_{23} = \{3, 4, 5, 6, 7\};$

$H_9 : U_1 = \{1\},\ U_2 = \{3\},\ U_3 = \{1, 7\},\ U_4 = \{2, 4\},\ U_5 = \{2, 5\},$
$U_6 = \{2, 6\},\ U_7 = \{3, 4\},\ U_8 = \{3, 5\},\ U_9 = \{3, 6\},$
$U_{10} = \{1, 2, 3\},\ U_{11} = \{1, 4, 7\},\ U_{12} = \{1, 5, 7\},\ U_{13} = \{1, 6, 7\},$
$U_{14} = \{2, 4, 5\},\ U_{15} = \{2, 4, 6\},\ U_{16} = \{2, 5, 6\},\ U_{17} = \{3, 4, 5, 7\},$
$U_{18} = \{3, 4, 6, 7\},\ U_{19} = \{3, 5, 6, 7\},\ U_{20} = \{2, 4, 5, 6, 7\},$
$U_{21} = \{3, 4, 5, 6, 7\},\ U_{22} = \{1, 2, 3, 4, 5, 6\};$

$H_{10}:\{j\},\ \{1,j\}\ (4\le j\le n),\ \{2\}\cup T,\ \{3\}\cup T$ if $n=7,\ \{1,2\}\cup T,$ $\{1,3\}\cup T$ if $n=11$, T is any set (possibly empty) of terminal vertices, $\{2,3\}\cup T_{n-5},\ T_{n-5}$ is any set of $n-5$ $(n\ge 5)$ terminal vertices, $\{1,2,3\}\cup T_{n-7},\ T_{n-7}$ is any set of $n-7$ $(n\ge 7)$ terminal vertices.

Sketch proof. For $i=1,2,\dots,9$ we can use an exhaustive search either by the computer, or by hand (based on Theorem 1.1; note, now the corresponding condition reads: $\mathbf{b}_u^T(\mu I-C(i))^{-1}\mathbf{b}_u=1$, where $C(i)$ is the adjacency matrix of H_i). In the case of H_{10}, we will demonstrate how we can get good sets by using only the Schwenk formula.

Consider first that U is the set of (any) k terminal vertices in H_{10}. Apply then (1) with $G=H_{10}(U)$ and $v=u$. Then we get

$$P_{H_{10}(U)}(1)=P_{H_{10}}(1)-\sum_{w\in U}P_{H_{10}-w}(1)-2\sum_{C\in\mathcal{C}(u)}P_{H_{10}(U)-C}(1).$$

Since $P_{H_{10}}(1)=-4$, $P_{H_{10}-w}(1)=-4$ $(w\in U)$, and $P_{H_{10}(U)-C}(1)=0$, $C\in\mathcal{C}(u)$, we have $P_{H_{10}(U)}(1)=-4+4k$, and therefore $P_{H_{10}(U)}(1)=0$ if and only if $k=1$. So, if U as a good set contains only terminal vertices of H_{10}, it is of the form $\{j\}$ $(4\le j\le n)$.

Consider next U contains at least one vertex from the triangle induced by the set $\{1,2,3\}$ and some terminal vertices determined by the set T, where T is possibly empty. Then, if we take that $\{1\}$, $\{2\}$, $\{3\}$, $\{1,2\}$, $\{1,3\}$, $\{2,3\}$ and $\{1,2,3\}$ are the vertices taken from the triangle we get that $P_{H_{10}(U)}(1)$ is equal to $-4+4k$, $-7+n$, $-7+n$, $-11+n$, $-11+n$, $-5-k+n$ and $-7-k+n$, respectively. Since $P_{H_{10}(U)}(1)=0$, all good sets are just those as required. This completes the proof.

We will now determine all H-maximal graphs (for 1) for some of the star complements H given in Theorem 2.1. As it is mentioned in Section 1, finding maximal graphs is equivalent to finding maximal cliques in the extendability graph. As already noted, a necessary and sufficient condition for u_1 and u_2 to be good partners follows from Theorem 1.1 (recall, if $\mathbf{b}_{u_1}$ and $\mathbf{b}_{u_2}$ are the characteristic vectors of U_1 and U_2, respectively then u_1 and u_2 are good partners if and only if ${\mathbf{b}_{u_1}}^T(\mu I-C)^{-1}\mathbf{b}_{u_2}$ is equal either 0 or -1). (Here, u_1 and u_2 are non–adjacent in the former case, and adjacent in the later case.) This easy criterion for checking if two good vertices are good partners will be used in sequel.

We first demonstrate this technique if H is equal to H_5 and H_6.

Theorem 2.3 *If H is equal to H_5 (or H_6) then there exists a unique H-maximal graph for* 1*.*

Proof. By Theorem 2.2, we have exactly four good sets in H_5. In this situation it is easy to check that each two of them are compatible. So there is a unique maximal graph which now arises. Similarly, we have exactly nine good sets in H_6. Again, we can check that each two of them are compatible. So there is a unique maximal graph which now arises. This completes the proof.

Remark 2.1 We now give some data for two H-maximal graphs (for 1) obtained in the previous theorem. The first graph is strongly regular (of degree 4) on 9 vertices. Its spectrum[a] is $[-2^4, 1^4, 4]$. The second graph is strongly regular (of degree 6) on 15 vertices. Its spectrum is $[-3^5, 1^9, 6]$.

In order to find all maximal graphs for a given star complement and an eigenvalue μ, one of the authors (Z.S.) has created an *SCL* (*star complement library*) – i.e. a collection of programs related to star complement technique. This library includes the programs for identifying good sets, for checking their compatibility, for finding maximal cliques and for identifying isomorphism classes. Some results obtained by making use of SCL facilities are given in the next theorem.

Theorem 2.4 *If H is one of the following graphs $H_1 - H_4$ and $H_7 - H_9$, then H-maximal graphs for* 1*, and some data about them (including the number of vertices and edges, spectrum, good sets (as denoted in Theorem 2.2)) are summarized below.*

H_1:
G_1 : 7, 12, $[-2, -1.65, -1^2, 1^2, 3.65]$; U_7, U_8.
G_2 : 8, 11, $[-2^2, -1^2, 1^3, 3]$; U_1, U_3, U_6.
G_3 : 10, 15, $[-2^4, 1^5, 3]$; U_1, U_2, U_3, U_4, U_5.

H_2:
G_4 : 6, 8, $[-2.29, -1, -0.60, 0, 1, 2.90]$; U_4.
G_5 : 8, 12, $[-3, -1^3, 1^3, 3]$; U_1, U_2, U_3.

H_3:
G_6 : 10, 20, $[-3, -2, -1.37, -1^2, 1^4, 4.37]$; U_8, U_{10}, U_{11}, U_{13}.
G_7 : 12, 24, $[-3.27, -2^3, -1, 1^6, 4.27]$; U_4, U_5, U_6, U_7, U_{12}, U_{14}.
G_8 : 16, 40, $[-3^5, 1^{10}, 5]$; U_1, U_2, U_3, U_4, U_5, U_6, U_7, U_8, U_{11}, U_{12}.

[a]In an exponential notation, an exponent stands for the multiplicity of the eigenvalue.

H_4:

$G_9:\ 10, 16, [-2.70, -2, -1^3, 1^4, 3.70];\ \ U_6,\ U_7,\ U_{10},\ U_{11}.$

$G_{10}: 12, 24, [-3.27, -2^3, -1, 1^6, 4.27];\ \ U_4,\ U_5,\ U_8,\ U_9,\ U_{12},\ U_{13}.$

$G_{11}: 16, 40, [-3^5, 1^{10}, 5];\ \ U_1,\ U_2,\ U_3,\ U_4,\ U_5,\ U_6,\ U_7,\ U_8,\ U_9,\ U_{12}.$

H_7:

$G_{13}: 10, 21, [-3, -2.46, -1^3, 1^4, 4.46];\ \ U_7,\ U_8,\ U_9,\ U_{10}.$

$G_{14}: 15, 45, [-3^5, 1^9, 6];\ \ U_1,\ U_2,\ U_3,\ U_4,\ U_5,\ U_6,\ U_7,\ U_8,\ U_{10}.$

H_8:

$G_{15}: 14,\ \ 37, [-4, -2.77, -2^2, -1^2, 1^7, 5.77];\ \ U_{10},\ U_{11},\ U_{12},\ U_{13},\ U_{14},\ U_{15},\ U_{16}.$

$G_{16}: 18,\ \ 57, [-4.66, -3^4, -1, 1^{11}, 6.66];\ \ U_4,\ U_5,\ U_6,\ U_7,\ U_8,\ U_9,\ U_{17},\ U_{18},\ U_{19},\ U_{20},\ U_{21}.$

$G_{17}: 27, 135, [-5^6, 1^{20}, 10];\ \ U_1,\ U_2,\ U_3,\ U_4,\ U_5,\ U_6,\ U_7,\ U_8,\ U_9,\ U_{10},\ U_{11},\ U_{12},\ U_{13},\ U_{14}, U_{17},\ U_{18},\ U_{19},\ U_{20},\ U_{22},\ U_{23}.$

H_9:

$G_{18}: 18,\ \ 57, [-4.66, -3^4, -1, 1^{11}, 6.66];\ \ U_7,\ U_8,\ U_9,\ U_{11},\ U_{12},\ U_{13},\ U_{14},\ U_{15},\ U_{16},\ U_{21},\ U_{22}.$

$G_{19}: 27, 135, [-5^6, 1^{20}, 10];\ \ U_1,\ U_2,\ U_3,\ U_4,\ U_5,\ U_6,\ U_7,\ U_8,\ U_9,\ U_{11},\ U_{12},\ U_{13},\ U_{14},\ U_{15},\ U_{16},\ U_{17},\ U_{18},\ U_{19},\ U_{20},\ U_{21}.$

Remark 2.2 Graphs G_3, G_8, G_{11}, G_{14}, G_{17} and G_{19} (with exactly three distinct eigenvalues) are all strongly regular graphs. In addition, G_3 is the Petersen graph; G_8 and G_{11} are equal to the Clebsch graph; G_{17} and G_{19} are equal to one of the Smith graphs (see [3]). We also note that the following pairs of graphs G_7 and G_{10}, G_8 and G_{11}, G_{16} and G_{18} and, G_{17} and G_{19} are isomorphic – so, they contain two different star complements, which are both unicyclic graphs.

Remark 2.3 According to [1] Theorem 3.1 (see also [7], p. 119) the following condition $n \le \frac{1}{2}\,t(t+1) - 1$ holds for any connected regular graph of order n, where t is the order of a star complement for an eigenvalue $\mu \notin \{-1, 0\}$. It is a remarkable fact that the Smith graph from the above remark (i.e. the graph G_{17}, or G_{19}) attains this bound.

Maximal graphs for H equal to H_{10} will be considered in our forthcoming research.

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