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A GAME BASED ON SPECTRAL GRAPH THEORY

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We present a mathematical game for two players based on spectral graph theory. We solve some cases and discuss a general strategy for unsolved ones. In addition, we present some data on graphs with integer index.

1. INTRODUCTION

We are focused on undirected graphs without loops or multiple edges, although the game, which we are going to describe, can be modified for the excluded cases, too. The eigenvalues of adjacency matrix of graph G are real and they form the *graph spectrum*. The largest eigenvalue is also called the *graph index* and it is usually denoted by r (or $r(G)$). As a consequence of the famous theorem of Frobenius from the matrix theory, the whole spectrum lies in the segment $[-r, r]$. The graph index is intensively studied (together or independently of the rest of spectrum) and there are many literature and papers in this topic (see, for instance, [**2**], [**4**]). Here we recall some of its properties which we will use in the future. Spectrum of a disconnected graph is the union of the spectra of its components. The index of a connected graph is greater than index of any of its proper induced subgraphs, while in the case of disconnected graphs the index, of course, has to be attained in some component. As another consequence of the mentioned theorem of Frobenius, the index of a connected graph is its simple eigenvalue, while a disconnected graph can have the index of a greater multiplicity. Therefore, inserting a new edge into a connected graph implies the increasing of its index; in the case of a disconnected graph the index may remain unchanged.

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Through the rest of text, the first and the second player are called F and S, respectively. Let G be graph on n vertices $v_1, v_2, \ldots, v_n$, without edges and let $m \in \mathbb{R}$ such that $1 \leq m < n-1$. Two players alternately insert one edge into G until index r becomes (strictly) greater than m. The player who has inserted the last edge loses the game. The question is: Which player has a winning strategy? (Obviously, for finite n, somebody wins after a sufficient number of moves.) The previous simple description of the game precedes to very complicated situation in general. However, there is a trivial case: $m = 1$. There F wins in cases $n = 3 + 4i$ and $n = 6 + 4i$, while S wins in cases $n = 4 + 4i$ and $5 + 4i$, $i = 0, 1, 2, \ldots$.

As we know, an eigenvalue of arbitrary graph is either integer or irrational number. Here, we focus our attention on the case when m is an integer, while the remaining situation can be a topic of some future research.

2. THE GAME FOR $m = 2$

Consider the case $m = 2$. In order to solve it, we need the of set graphs with property $r = 2$. These graphs are known as SMITH graphs (see [**5**]) and they are depicted on Figure 1. Note that the first of displayed graphs is known as a C_n, while in the second graph $l \geq 0$ denotes the length of the corresponding path and for $\ell = 0$ this graph reduces to $K_{1,4}$. It is known that every graph with index $r < 2$ is contained in some SMITH graph, every connected graph with index $r = 2$ is one of the SMITH graphs and every graph with index $r > 2$ contains some SMITH graph. This nice property of SMITH graphs enables us to prove the following theorem.

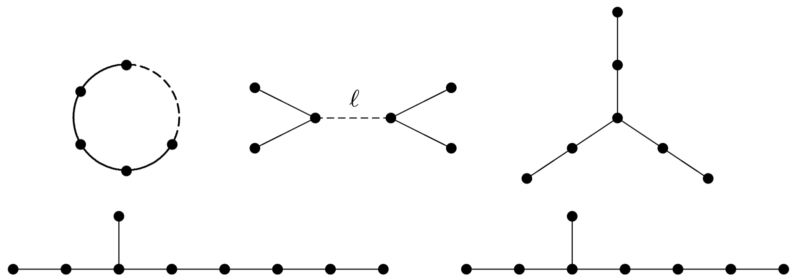

Figure 1.

Theorem 1. *In case $m = 2$, F has the winning strategy in cases $n = 6 + 4i$ and $n = 7 + 4i$, while S has the winning strategy in cases $n = 4 + 4i$ and $n = 5 + 4i$, $i = 0, 1, 2, \ldots$.*

Proof. It is easy to prove the statement in case $i = 0$ (by hand, or by using the computer). Suppose that the statement holds for all integers less than i and prove it for i. Without loosing the generality, suppose that F in his first move inserts the edge v_1v_2.

Case 1: $n = 4 + 4i$. If S plays v_2v_3, he directly loses the game (F makes a cycle (SMITH graph) of vertices v_1, v_2 and v_3, and wins the game on the rest of the graph by hypothesis.) Therefore, S has to play v_3v_4. Now, we have 3 possible moves for F (up to isomorphism): v_2v_3, v_4v_5 and v_5v_6. On v_2v_3, S plays v_1v_4 and wins by hypothesis. On v_4v_5, S plays v_2v_4. Now, there are 4 possible moves for F: v_2v_6 (with this move F has just formed SMITH graph and he loses by hypothesis), v_1v_6, v_3v_6 and v_6v_7 (in these 3 cases S forms the SMITH graph by v_1v_7, v_5v_7 and v_2v_8, respectively and he wins by hypothesis). The remaining case is when F plays v_5v_6 in his second move. On that, S plays v_7v_8 and so on. Because of $n = 4 + 4i$, F is the one who (before or later) must join some vertex of degree $d > 0$ to some other vertex. It is an easy exercise to prove that he loses after that move (similarly as in previous consideration).

Case 2: $n = 5+4i$. After S's move v_2v_3, F has 4 possible moves: v_1v_3 (this forming of SMITH graph leads directly to defeat), v_2v_4, v_3v_4 and v_4v_5 (in all cases S forms the SMITH graph and wins, by v_2v_5, v_1v_4 and v_1v_3, respectively).

Case 3: $n = 6 + 4i$. S has 2 possible moves: v_2v_3 and v_3v_4 (F's answer to one of these moves is the other one, which implies the same situation in both cases after his second move). Now, if S forms the SMITH graph he loses, otherwise F will form SMITH graph in his next move and S is defeated, again.

Case 4: $n = 7 + 4i$. In the most complicated situation F wins with S's strategy from the end of Case 1. Therefore, the proof will be omitted as an easy exercise, again. The remaining situations are simple.

This completes the proof.

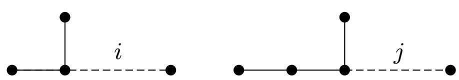

Figure 2.

For given order n and integer m, $(1 \le m \le n-2)$ let us consider the set of maximal graphs of order n, with property $r \le m$ (i. e. index r becomes strictly greater than m by inserting an arbitrary edge). Note that these graphs do not have to be connected. Denote the set of these graphs by $\mathcal{M}_{n,m} = \{G_1, G_2, \dots, G_k\}$. Hence, the final goal of our game can be preformulated in the following way: the looser is the player who first has made a proper supergraph of one of graphs from the set $\mathcal{M}_{n,m}$. Therefore, the mentioned set plays a crucial role in game, because the interest of each player is to construct one of its members G_i, $i \in \{1, 2, \dots k\}$ in order to force the other player to form a proper supergraph of G_i.

Note that the set $\mathcal{M}_{n,m}$ can be partitioned in two subsets: $\mathcal{M}^{=}_{n,m} = \{G : G \in \mathcal{M}_{n,m},\ r(G) = m\}$ and $\mathcal{M}^{<}_{n,m} = \{G : G \in \mathcal{M}_{n,m},\ r(G) < m\}$. Unfortunately, mentioned set is easily describable only in the case $m = 2$. The following two lemmas hold.

Lemma 1. *A connected graph of order n belongs to set $\mathcal{M}^{<}_{n,2}$ if and only if it has a form as on Figure* 2 (*where i and j denote the length of the corresponding paths and $i \ge 1$, $j \in \{2, 3, 4\}$ hold.*)

Proof. Each of these graphs is a proper subgraph of some SMITH graph. This implies two properties of such graph: (1) it is a tree; (2) there is exactly one vertex of degree 3 and every other vertex has degree 1 or 2. By using this properties, we easily obtain displayed family.

Lemma 2. *A graph G of order n belongs to $\mathcal{M}_{n,2}$ if and only if every its component is one of graphs displayed on Figure* 1 *or Figure* 2, *where at most one component is isomorphic to the first graph of Figure* 2.

Proof. This statement is an immediate consequence of definition of the set $\mathcal{M}_{n,2}$, well known properties of SMITH graphs and the previous lemma. Two components cannot be isomorphic to the first graph of Figure 2, because of maximality of graph G. Namely, two such components can be joined by an edge such that they become one of SMITH graphs.

3. SOME COMPUTATIONAL RESULTS

The maximal graphs with property $r \leq 2$ enable us to treat the case $m = 2$ (even less, we prove Theorem 1 by using the SMITH graphs). Unfortunately, there is no some general description of maximal graphs in other cases. This fact constrains our research on graphs with small order and force us to use a computer in our research. In this way, we obtain the sets $\mathcal{M}_{n,m}$ and solve some cases of the game by using them. Computational results in these cases are given in the following table. Three data (the cardinality of set $\mathcal{M}_{n,m}$, the number of members of $\mathcal{M}_{n,m}$ which have odd order and the winner of the game) are given in every field.

$n \backslash m$	3	4	5
5	4 (3), F	-	-
6	16 (4), S	9 (7), F	-
7	54 (35), F	71 (61), F	9 (8), F

n	8	9	10
$m = n - 2$	25 (24), F	25 (10), ?	66 (61), F

Table 1.

We find that the size of maximal graphs from the set $\mathcal{M}_{n,m}$ has an important role. For instance, in case $n = 8$, $m = 6$ only one maximal graph has an even number of edges which gives us a reason to suppose that here player F has a winning strategy and later we succeeded in proving this conjecture. We use a similar reasoning in other cases. Note that situation quickly becomes too complicated even for a computer (in sense that it needs much time for solving it, for example, in the case $n = 9$, $m = 7$, which remains open). Clearly, some particular cases can be solved, but, by author's opinion, it is hard to believe that the game can be solved in general. However, some future results in the theory of graph spectra and advances in computer technology should help in obtaining new results.

Let us finish this section with some remarks about the general strategy. Suppose that we obtain some graph $\bar{G}$ after last move of one player. Then, we can define the sets:

$\mathcal{G}^F_{n,\,m}(\bar{G}) = \{G : G \in \mathcal{M}_{n,\,m},\ |E(G)|$ is odd and $\bar{G}$ is a proper subgraph of $G\}$,

$\mathcal{G}^S_{n,\,m}(\bar{G}) = \{G : G \in \mathcal{M}_{n,\,m},\ |E(G)|$ is even and $\bar{G}$ is a proper subgraph of $G\}$.

If the ratio $\dfrac{|\mathcal{G}^F_{n,\,m}(\bar{G})|}{|\mathcal{G}^S_{n,\,m}(\bar{G})|}$ increases during the game, the possibility of F's win increases and viceversa (see Table 1 for the ratio at the beginning of the game). Hence, F could have the following strategy: playing the move such that mentioned ratio becomes as big as possible after this move. Furthermore, by including the calculation in depth, he could play a move such that the ratio becomes biggest possible after the next k moves. Obviously, this strategy requires huge calculations and can be applied only with aid of computer.

4. SOME ADDITIONAL DATA

In the previous sections we deal with the sets $\mathcal{M}^{=}_{n,\,m}$ and $\mathcal{M}^{<}_{n,\,m}$. The first of these sets contains graphs with integer index and we find that there are not very many such graphs (up to 10 vertices) which are connected. We give the data on these graphs in Table 2.

Two data (the number of graphs with given order and index, and number of such graphs which are regular) are given in every field. (More on regular graphs one can find on http://www.mathe2.uni-bayreuth.de/markus/reggraphs.html.)

There is only one connected graph with index 1 and 17 connected graphs with index 2 up to order 10 (compare Figure 1). If we add these values to the sum according to Table 2, we obtain exactly 1328 connected graphs with integer index up to 10 vertices. Numbers of regular graphs are well known, while the data for non–regular case are new. These graphs are available on the following address http://www.matf.bg.ac.yu/∼zstanic/indexdiam.html.

$n\backslash r$	3	4	5	6	7	8	9
4	1 (1)	-	-	-	-	-	-
5	1 (0)	1 (1)	-	-	-	-	-
6	2 (2)	1 (1)	1 (1)	-	-	-	-
7	5 (0)	3 (2)	1 (0)	1 (1)	-	-	-
8	18 (5)	20 (6)	8 (3)	1 (1)	1 (1)	-	-
9	36 (0)	103 (16)	18 (0)	11 (4)	1 (0)	1 (1)	-
10	135 (19)	582 (59)	256 (60)	91 (21)	8 (5)	2 (1)	1 (1)

Table 2.

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